Math SSRU

Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Final Examination
Semester 2/2022

Course ID	Course Name	Test Time	Full Scores
MAC3309	Mathematical Analysis	1pm - 4pm Tue 28 Mar 2023	100 marks 25%

Name.. **ID**................................... **Section**...............

Direction

1. 10 questions of all 10 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
4. Cannot answer sheets out of test room.
5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

1. **(10 marks)** Use definition to prove that

$$f(x) = x^3 + 1$$

is continuous at $x = 1$.

2. **(10 marks)** Let $f : [0, 1] \to \mathbb{R}$ be uniformly continuous on $[0, 1]$. Assume that

$$f + g \text{ is uniformly continuous on } [0, 1].$$

Prove that g is uniformly continuous on $[0, 1]$.

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\sqrt{x} \leq x \quad \text{for all} \quad x \geq 1.$$

4. **(10 marks)** Let $f(x) = e^x - e^{-x}$ where $x \in \mathbb{R}$.

   4.1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

   4.2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

   4.3 **(3 marks)** Compute $(f^{-1})'(\frac{3}{2})$.

5. **(10 marks)** Define

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2 & \text{if } x \in (0,1) \cup (1,2) \end{cases}$$

Use definition to show that f is integrable on $[0, 2]$

6. **(10 marks)** Let $f(x) = 2x + 1$ where $x \in [0, 1]$ and

$$P = \left\{ \frac{2j+1}{2n+1} : j = 0, 1, ..., n \right\} = \left\{ \frac{1}{2n+1}, \frac{3}{2n+1}, \frac{5}{2n+1}, ..., 1 \right\}$$

be a partition of $[0, 1]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$.

7. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} \frac{g(t)}{\sqrt{t}}\,dt \quad \text{where } x > 0.$$

Show that $\displaystyle\int_0^1 f(x) + g(x)\,dx = 0$.

Hint: Use integration by part to $\displaystyle\int_0^1 f(x)\,dx$.

8. **(10 marks)** Evaluate the infinite sum :

$$\sum_{k=1}^{\infty} \frac{1}{2023^k} \left[2022 - \frac{1}{2023^{k-1}} + \frac{1}{2023^{k+1}} \right].$$

Hint: Use Telescoping and Geometric Series.

9. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty}(a_k + b_k) \text{ converges absolutely.}$$

Hint: Use Cauchy criterion

10. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k}\right)$$

is conditionally convergent.

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Solution Final Exam. 2/2022
MAC3309 Mathematical Analysis

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1. **(10 marks)** Use definition to prove that

$$f(x) = x^3 + 1$$

is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{7}\}$ such that $|x-1| < \delta$. Then $|x-1| < 1$. So,

$$|x| - 1 \leq |x-1| < 1.$$

Thus, $|x| < 2$ and $|x^2| < 4$. We obtain

$$\begin{aligned} |f(x) - f(1)| &= |(x^3+1) - 2| = |x^3 - 1| \\ &= |(x-1)(x^2+x+1)| = |x-1||x^2+x+1| \\ &< \delta(|x|^2 + |x| + 1) < \frac{\varepsilon}{7} \cdot (4+2+1) = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 1$. □

2. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Assume that

$$f + g \text{ is uniformly continuous on } [0,1].$$

Prove that g is uniformly continuous on $[0,1]$.

Proof. Assume that f and $f+g$ be uniformly continuous on $[0,1]$.
Let $\varepsilon > 0$. There is an $\delta_1 > 0$ such that

$$|x-a| < \delta_1 \text{ for all } x, a \in [0,1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

and there is an $\delta_2 > 0$ such that

$$|x-a| < \delta_2 \text{ for all } x, a \in [0,1] \quad \text{implies} \quad |f(x) + g(x) - f(a) - g(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $x, a \in [0,1]$ such that $|x-a| < \delta$. Apply the triangle inequality, we have

$$|g(x) - g(a)| - |f(x) - f(a)| < |f(x) + g(x) - f(a) - g(a)| < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} |g(x) - g(a)| &= \frac{\varepsilon}{2} + |f(x) - f(a)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, g is uniformly continuous on $[0,1]$. □

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\sqrt{x} \leq x \quad \text{for all} \quad x \geq 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \sqrt{x} - x \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$\begin{aligned} f(1) &= 0 \\ f'(x) &= \frac{1}{2\sqrt{x}} - 1 \end{aligned}$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a - 1) \\ \sqrt{a} - a &= \left(\frac{1}{2\sqrt{c}} - 1\right)(a - 1) \end{aligned}$$

From $c > 1$, it leads to $\sqrt{c} > 1$ or $2\sqrt{c} > 2 > 1$. So, $\frac{1}{2\sqrt{c}} < 1$. We have

$$\frac{1}{2\sqrt{c}} - 1 < 0.$$

Since $a > 1$, $a - 1 > 0$. Therefore,

$$\sqrt{a} - a = \left(\frac{1}{2\sqrt{c}} - 1\right)(a - 1) < 0$$

Therefore, We conclude that $\sqrt{x} \leq x \quad \text{for all} \quad x \geq 1.$ □

4. **(10 marks)** Let $f(x) = e^x - e^{-x}$ where $x \in \mathbb{R}$.

4.1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $-x < -y$. We obtain

$$e^x > e^y \text{ and } e^{-x} < e^{-y}.$$

So, $-e^{-x} > -e^{-y}$. It follows that

$$\begin{gathered} e^x - e^{-x} > e^y - e^{-y} \\ f(x) > f(y) \end{gathered}$$

So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

4.2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

4.3 **(3 marks)** Compute $(f^{-1})'(\frac{3}{2})$.

Solution. We see that $f'(x) = e^x + e^{-x}$ and

$$f(\ln 2) = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2}.$$

So $f^{-1}(\frac{3}{2}) = \ln 2$. By IFT,

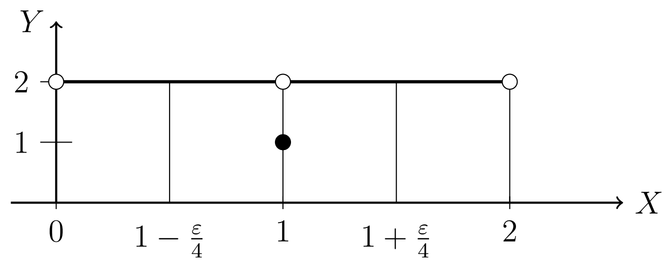
$$\begin{aligned} (f^{-1})'\left(\frac{3}{2}\right) &= \frac{1}{f'(f^{-1}(\frac{3}{2}))} \\ &= \frac{1}{f'(\ln 2)} \\ &= \frac{1}{e^{\ln 2} + e^{-\ln 2}} \\ &= \frac{1}{2 + \frac{1}{2}} \\ &= \frac{2}{5} \quad \# \end{aligned}$$

5. **(10 marks)** Define

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2 & \text{if } x \in (0,1) \cup (1,2) \end{cases}$$

Use definition to show that f is integrable on $[0,2]$

Solution. Let $\varepsilon > 0$. Case $\varepsilon \leq 2$. Choose $P = \left\{0, 1 - \frac{\varepsilon}{4}, 1, 1 + \frac{\varepsilon}{4}, 2\right\}$.



We obtain

$$U(f,P) = 2\left(1 - \frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) + 2\left(1 - \frac{\varepsilon}{4}\right)$$
$$L(f,P) = 2\left(1 - \frac{\varepsilon}{4}\right) + 1\left(\frac{\varepsilon}{4}\right) + 1\left(\frac{\varepsilon}{4}\right) + 2\left(1 - \frac{\varepsilon}{4}\right)$$
$$U(f,P) - L(f,P) = \frac{\varepsilon}{2} < \varepsilon.$$

Case $\varepsilon > 2$. Choose $P = \{0, 1, 2\}$. Then

$$U(f,P) = 2(1-0) + 2(2-1)$$
$$L(f,P) = 1(1-0) + 1(2-1)$$
$$U(f,P) - L(f,P) = 2 < \varepsilon.$$

Thus, f is integrable on $[0,2]$.

6. **(10 marks)** Let $f(x) = 2x + 1$ where $x \in [0, 1]$ and

$$P = \left\{ \frac{2j+1}{2n+1} : j = 0, 1, ..., n \right\} = \left\{ \frac{1}{2n+1}, \frac{3}{2n+1}, \frac{5}{2n+1}, ..., 1 \right\}$$

be a partition of $[0, 1]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$.

Solution. Choose **The Right End Point** , i.e., $f(t_j) = f(\frac{2j+1}{2n+1})$ on the subinterval $[x_{j-1}, x_j]$ and

$$\Delta x_j = \frac{2j+1}{2n+1} - \frac{2(j-1)+1}{2n+1} = \frac{2}{2n+1} \qquad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{2j+1}{2n+1}\right)\frac{2}{2n+1} = \frac{2}{2n+1}\sum_{j=1}^{n}\left[2\left(\frac{2j+1}{2n+1}\right)+1\right] \\
&= \frac{2}{2n+1}\left[\frac{2}{2n+1}\sum_{j=1}^{n}(2j+1) + \sum_{j=1}^{n} 1\right] \\
&= \frac{2}{2n+1}\left[\frac{2}{2n+1}\left(2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1\right) + n\right] \\
&= \frac{2}{2n+1}\left[\frac{2}{2n+1}\left(2\cdot\frac{n(n+1)}{2} + n\right) + n\right] \\
&= \frac{2}{2n+1}\left[\frac{2(n^2+2n)}{2n+1} + n\right] \\
&= \frac{4(n^2+2n)}{(2n+1)^2} + \frac{2n}{2n+1}
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{4(n^2+2n)}{(2n+1)^2} + \frac{2n}{2n+1} = 1 + 1 = 2 \quad \#$$

7. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} \frac{g(t)}{\sqrt{t}}\, dt \quad \text{where } x > 0.$$

Show that $\int_0^1 f(x) + g(x)\, dx = 0$.

Hint: Use integration by part to $\int_0^1 f(x)\, dx$.

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = \frac{g(x^2)}{\sqrt{x^2}} \cdot 2x = 2g(x^2).$$

By integration by part, we obtain

$$\begin{aligned}
\int_0^1 f(x)\, dx &= \int_0^1 x' f(x)\, dx = [xf(x)]_0^1 - \int_0^1 xf'(x)\, dx \\
&= 1f(1) - 0f(0) - \int_0^1 x \cdot 2g(x^2)\, dx \\
&= f(1) - \int_0^1 g(x^2)(x^2)'\, dx \\
&= \int_1^1 \frac{g(t)}{\sqrt{t}}\, dt - \int_0^1 g(\phi(x))\phi'(x)\, dx \qquad \text{Change of Variable } \phi(x) = x^2 \\
&= 0 - \int_{\phi(0)}^{\phi(1)} g(t)\, dt \\
&= -\int_0^1 g(t)\, dt \\
&= -\int_0^1 g(x)\, dx.
\end{aligned}$$

Thus, $\int_0^1 f(x) + g(x)\, dx = \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx = 0$.

8. **(10 marks)** Evaluate the infinite sum :

$$\sum_{k=1}^{\infty} \frac{1}{2023^k} \left[2022 - \frac{1}{2023^{k-1}} + \frac{1}{2023^{k+1}} \right].$$

Hint: Use Telescoping and Geometric Series.

Solution. Consider

$$\begin{aligned} \frac{1}{2023^k} \left[2022 - \frac{1}{2023^{k-1}} + \frac{1}{2023^{k+1}} \right] &= 2022 \cdot \frac{1}{2023^k} - \frac{1}{2023^{2k-1}} + \frac{1}{2023^{2k+1}} \\ &= 2022 \cdot \frac{1}{2023^k} - \left(\frac{1}{2023^{2k-1}} - \frac{1}{2023^{2k+1}} \right). \end{aligned}$$

We obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{2023^k} \left[2022 - \frac{1}{2023^{k-1}} + \frac{1}{2023^{k+1}} \right] &= \sum_{k=1}^{\infty} \left[2022 \cdot \frac{1}{2023^k} - \left(\frac{1}{2023^{2k-1}} - \frac{1}{2023^{2k+1}} \right) \right] \\ &= 2022 \sum_{k=1}^{\infty} \frac{1}{2023^k} - \sum_{k=1}^{\infty} \left(\frac{1}{2023^{2k-1}} - \frac{1}{2023^{2k+1}} \right). \\ &= 2022 \cdot \frac{\frac{1}{2023}}{1 - \frac{1}{2023}} - \left(\frac{1}{2023} - \lim_{k \to \infty} \frac{1}{2023^{2k+1}} \right) \\ &= 1 - \frac{1}{2023} = \frac{2022}{2023} \quad \# \end{aligned}$$

9. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty}(a_k + b_k) \text{ converges absolutely.}$$

Hint: Use Cauchy criterion

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converges absolutely. Then $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} |b_k|$ converge. Let $\varepsilon > 0$. By Cauchy criterion, there is an $N_1 \in \mathbb{N}$ such that

$$m > n \geq N_1 \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \frac{\varepsilon}{2}.$$

and there is an $N_2 \in \mathbb{N}$ such that

$$m > n \geq N_2 \quad \text{implies} \quad \sum_{k=n}^{m} |b_k| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. We obatin

$$\begin{aligned}
\sum_{k=n}^{m} |a_k + b_k| &\leq \sum_{k=n}^{m}(|a_k| + |b_k|) \\
&= \sum_{k=n}^{m} |a_k| + \sum_{k=n}^{m} |b_k| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\sum_{k=1}^{\infty} |a_k + b_k|$ converges. This result concluded that $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.

□

10. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k}\right)$$

is conditionally convergent.

Solution. Firstly, we see that

$$\lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right) = 0.$$

Next, let $f(x) = \ln\left(1+\frac{1}{x}\right)$ where $x \geq 1$. The derivative of $f(x)$ is

$$f'(x) = \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) < 0 \quad \text{for all } x \geq 1.$$

So, $\left\{\ln\left(1+\frac{1}{k}\right)\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k}\right) \quad \text{converges.}$$

Finally, we consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \ln\left(1+\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$$

and

$$\lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{1}{1+\frac{1}{k}} \cdot \left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges ($p = 1$), by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right) \quad \text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k}\right) \quad \text{is conditionally convergent.}$$