Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Final Examination
Semester 2/2023

Course ID	Course Name	Test Time	Full Scores
MAC3309	Mathematical Analysis	9 a.m. - 12 a.m. Wed 27 Mar. 2024	100 marks 25%

Name.. **ID**.................................... **Section**...............

Direction

1. 10 questions of all 12 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
4. Cannot answer sheets out of test room.
5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

Some Definition to prove this examination.

1. f is continuous at a $\iff$ $\forall\varepsilon > 0\ \exists\delta > 0,\ |x-a| < \delta \longrightarrow |f(x)-f(a)| < \varepsilon$

2. f is uniformly continuous on E $\iff$ $\forall\varepsilon > 0\ \exists\delta > 0\ \forall x, a \in E,\ |x-a| < \delta \longrightarrow |f(x)-f(a)| < \varepsilon$

3. f is differentiable at a $\iff$ $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists

4. f is increasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \ \longrightarrow\ f(x_1) < f(x_2)$

5. f is decreasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \ \longrightarrow\ f(x_1) > f(x_2)$

6. f is integrable on $[a,b]$ $\iff$ $\forall\varepsilon > 0\ \exists P_\varepsilon,\ U(f,P) - L(f,P) < \varepsilon$

7. Upper integral $\displaystyle (U)\int_a^b f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$

8. Lower integral $\displaystyle (L)\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$

9. Riemann sum converges to $I(f)$ $\iff$ $\displaystyle \forall\varepsilon > 0\ \exists P_\varepsilon \subseteq \{x_0, x_1, ..., x_n\} \longrightarrow \left|\sum_{i=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \varepsilon$

10. **Cauchy Criterion**: $\displaystyle\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\displaystyle \forall\varepsilon > 0\ \exists N \in \mathbb{N}, m > n \geq N \longrightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

$\iff$ $\displaystyle \forall\varepsilon > 0\ \exists N \in \mathbb{N}, m > n \geq N \longrightarrow \sum_{k=n}^{m} |a_k| < \varepsilon$

1. **(10 marks)** Use definition to prove that

$$f(x) = (x-1)(x+1) + 24$$

is continuous at $x = -1$.

2. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define

$$g(x) = xf(x) \quad \text{where } x \in [0,1].$$

Prove that g is uniformly continuous on $[0,1]$.
Hint : Use Extreme Value Theorem (EVT), i.e., if f is continous on E, then $\exists M > 0$ such that

$$|f(x)| \leq M \quad \text{for all } x \in E.$$

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x \leq \sqrt{x} \quad \text{for all} \quad x \geq 1.$$

4. **(10 marks)** Define $f(x) = x + \ln x$ where $x \in \mathbb{R}^+$.

   4.1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}^+$.

   4.2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}^+$.

   4.3 **(3 marks)** Compute $(f^{-1})'(1)$.

5. **(10 marks)** Define

$$f(x) = \begin{cases} 2 & \text{if } x \in (0,1) \\ 1 & \text{if } x \in [1,2). \end{cases}$$

Draw the graph of f on $[0,2]$ and use definition to show that f is integrable on $[0,2]$.

6. **(10 marks)** Let $f(x) = (x-1)(x+1) + 24$ where $x \in [0, 2]$ and

$$P = \left\{ \frac{2j}{n} : j = 0, 1, ..., n \right\} = \left\{ 0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, ..., 2 \right\}$$

be a partition of $[0, 2]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$ on $[0, 2]$.

7. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} 2\sqrt{t} \cdot g(t^2)\, dt \quad \text{where } x \in \mathbb{R}.$$

Show that $\displaystyle\int_{-1}^{0} f(x)\, dx + \int_0^1 g(x)\, dx = 0.$

Hint: Use integration by part to $\displaystyle\int_{-1}^{0} f(x)\, dx$ and change variable.

8. **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty} \frac{1}{a^k} \left[1 - \frac{1}{(k+1)a} + \frac{1}{k}\right] = \frac{1}{132}.$$

Hint: Use Telescoping and Geometric Series.

9. **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \Big(a + (-1)^k\Big)^k \text{ converges or NOT.}$$

Verify your answer.

10. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{k+1}{k^2+1}\right)$$

is conditionally convergent.

Solution Final Exam. 2/2023
MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

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1. **(10 marks)** Use definition to prove that

$$f(x) = (x-1)(x+1) + 24$$

is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{3}\}$.
Let $x \in \mathbb{R}$ such that $|x+1| < \delta$. Then $|x+1| < 1$.

$$\text{So, } |x| - |1| \le |x+1| < 1. \text{ We obtain } |x| \le 2.$$

By triangle inequility, it follows that

$$\begin{aligned} |f(x) - f(-1)| &= |(x-1)(x+1) + 24 - 24| \\ &= |x-1||x+1| \\ &< (|x|+1)\delta \\ &< (2+1)\delta \\ &= 3\delta < 3 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = -1$. □

2. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define

$$g(x) = xf(x) \quad \text{where } x \in [0,1].$$

Prove that g is uniformly continuous on $[0,1]$.
Hint : Use Extreme Value Theorem (EVT), i.e., if f is continous on E, then $\exists M > 0$ such that

$$|f(x)| \le M \quad \text{for all } x \in E.$$

Proof. Assume that f be uniformly continuous on $[0,1]$.
Let $\varepsilon > 0$. There is an $\delta_0 > 0$ such that

$$|x-a| < \delta_0 \text{ for all } x, a \in [0,1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Since f is continuous on $[0,1]$, by EVT, there is an $M > 0$ such that

$$|f(x)| \le M \quad \text{for all } x \in [0,1].$$

Choose $\delta = \min\left\{\delta_0, \dfrac{\varepsilon}{2(M+1)}\right\}$. Let $x, a \in [0,1]$ such that $|x-a| < \delta$. Then $|x| \le 1$ and $|f(a)| \le M$.
Apply the triangle inequality, we have

$$\begin{aligned}
|g(x) - g(a)| &= |xf(x) - af(a)| \\
&= |xf(x) - af(a) + xf(a) - xf(a)| \\
&= |x[f(x) - f(a)] + f(a)[x-a]| \\
&\le |x| \cdot |f(x) - f(a)| + |f(a)| \cdot |x-a| \\
&< 1 \cdot \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{2(M+1)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot 1 = \varepsilon.
\end{aligned}$$

Thus, g is uniformly continuous on $[0,1]$. □

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x \le \sqrt{x} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \sqrt{x} - \ln x \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 1$$
$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x}$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a-1) \\ \sqrt{a} - \ln a - 1 &= \left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right)(a-1) \\ \sqrt{a} - \ln a &= \left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right)(a-1) + 1 \end{aligned}$$

From $1 < c < a$, it leads to $1 < \sqrt{c} < \sqrt{a}$ and

$$\begin{array}{ccccccccccc} \frac{1}{a} & < & \frac{1}{c} & < & 1 & \longrightarrow & -1 & < & -\frac{1}{c} & < & -\frac{1}{a} \\ 2 & < & 2\sqrt{c} & < & 2\sqrt{a} & \longrightarrow & \frac{1}{2\sqrt{a}} & < & \frac{1}{2\sqrt{c}} & < & \frac{1}{2} \end{array}$$

We have $\left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right) < \left(-\frac{1}{a} + \frac{1}{2}\right)$. Since $a > 1$, $a - 1 > 0$ and $a(a-1) > 0$ and

$$\begin{aligned} \left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right)(a-1) &> \left(-\frac{1}{a} + \frac{1}{2}\right)(a-1) \\ \left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right)(a-1) + 1 &> \left(-\frac{1}{a} + \frac{1}{2}\right)(a-1) + 1 \\ &= -1 + \frac{1}{a} + \frac{a}{2} - \frac{1}{2} + 1 \\ &= \frac{1}{a} + \frac{a}{2} - \frac{1}{2} \\ &= \frac{2 + a^2 - a}{2a} \\ &= \frac{a(a-1) + 2}{2a} > 0 \end{aligned}$$

Thus,

$$\sqrt{a} - \ln a = \left(\frac{1}{2\sqrt{c}} - \frac{1}{c}\right)(a-1) + 1 > 0$$

We conclude that $\ln x \le \sqrt{x}$ for all $x \ge 1$. □

4. **(10 marks)** Define $f(x) = x + \ln x$ where $x \in \mathbb{R}^+$.

4.1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}^+$.

Proof. Let $x, y \in \mathbb{R}^+$ and $x \neq y$. WLOG $x > y > 0$. We obtain

$$\ln x > \ln y.$$

It follows that

$$\begin{aligned} x + \ln x &> y + \ln y \\ f(x) &> f(y) \end{aligned}$$

So, $f(x) \neq f(y)$. Therefore, f is injective on $\mathbb{R}^+$. □

4.2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}^+$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}^+$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}^+$.

4.3 **(3 marks)** Compute $(f^{-1})'(1)$.

Solution. We see that $f'(x) = 1 + \dfrac{1}{x}$ and

$$f(1) = 1 + \ln 1 = 1 + 0 = 1.$$

So $f^{-1}(1) = 1$. By IFT,

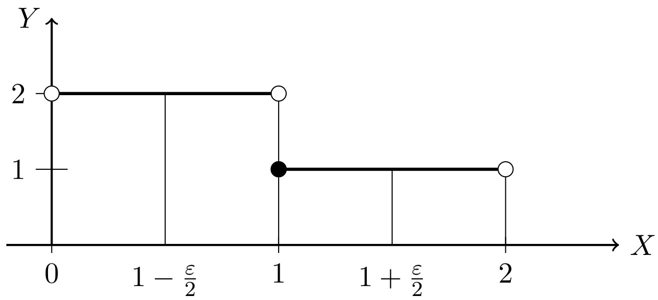
$$\begin{aligned} (f^{-1})'(1) &= \frac{1}{f'(f^{-1}(1))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{1+\frac{1}{1}} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \quad \# \end{aligned}$$

5. **(10 marks)** Define

$$f(x) = \begin{cases} 2 & \text{if } x \in (0,1) \\ 1 & \text{if } x \in [1,2). \end{cases}$$

Draw the graph of f on $[0,2]$ and use definition to show that f is integrable on $[0,2]$.

Proof. Let $\varepsilon > 0$.
Case $\varepsilon \le 1$. Choose $P = \left\{0, 1-\frac{\varepsilon}{2}, 1, 1+\frac{\varepsilon}{2}, 2\right\}$.



We obtain

$$\begin{aligned} U(f,P) &= 2\left(1-\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(1-\frac{\varepsilon}{2}\right) \\ L(f,P) &= 2\left(1-\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(1-\frac{\varepsilon}{2}\right) \\ U(f,P) - L(f,P) &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Case $\varepsilon > 1$. Choose $P = \{0,1,2\}$. Then

$$\begin{aligned} U(f,P) &= 2\,(1-0) + 1\,(2-1) \\ L(f,P) &= 1\,(1-0) + 1\,(2-1) \\ U(f,P) - L(f,P) &= 1 < \varepsilon. \end{aligned}$$

Thus, f is integrable on $[0,2]$. □

6. **(10 marks)** Let $f(x) = (x-1)(x+1) + 24$ where $x \in [0,2]$ and

$$P = \left\{ \frac{2j}{n} : j = 0, 1, ..., n \right\} = \left\{ 0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, ..., 2 \right\}$$

be a partition of $[0,2]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$ on $[0,2]$.

Solution. Choose **The Right End Point** , i.e., $f(t_j) = f(\frac{2j}{n})$ on the subinterval $[x_{j-1}, x_j]$ and

$$\Delta x_j = \frac{2j}{n} - \frac{2(j-1)}{n} = \frac{2}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

From $f(x) = (x-1)(x+1) + 24 = x^2 - 1 + 24 = x^2 + 23$. We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{2j}{n}\right)\frac{2}{n} = \frac{2}{n}\sum_{j=1}^{n}\left[\left(\frac{2j}{n}\right)^2 + 23\right] \\
&= \frac{2}{n}\left[\sum_{j=1}^{n}\frac{4j^2}{n^2} + \sum_{j=1}^{n} 23\right] \\
&= \frac{2}{n}\left[\frac{4}{n^2}\sum_{j=1}^{n} j^2 + 23n\right] \\
&= \frac{2}{n}\left[\frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + 23n\right] \\
&= \frac{4(n+1)(2n+1)}{3n^2} + 46
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{4(n+1)(2n+1)}{3n^2} + 46 = \frac{8}{3} + 46 = \frac{146}{3} \quad \#$$

7. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} 2\sqrt{t} \cdot g(t^2)\, dt \quad \text{where } x \in \mathbb{R}.$$

Show that $\int_{-1}^{0} f(x)\, dx + \int_0^1 g(x)\, dx = 0$.

Hint: Use integration by part to $\int_{-1}^{0} f(x)\, dx$ and change variable.

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = 2\sqrt{x^2} \cdot g((x^2)^2) \cdot 2x = 2|x| \cdot g(x^4) \cdot 2x = 4x|x| \cdot g(x^4).$$

We have

$$f(-1) = \int_1^1 2\sqrt{t} \cdot g(t^2)\, dt = 0$$

By integration by part, we obtain

$$\begin{aligned}
\int_{-1}^{0} f(x)\, dx &= \int_{-1}^{0} x' f(x)\, dx = [xf(x)]_{-1}^{0} - \int_{-1}^{0} xf'(x)\, dx \\
&= 0f(0) - (-1)f(-1) - \int_{-1}^{0} x \cdot 4x|x| g(x^4)\, dx \\
&= 0 - 0 - \int_{-1}^{0} x \cdot 4x(-x) g(x^4)\, dx \\
&= \int_{-1}^{0} 4x^3 g(x^4)\, dx \\
&= \int_{-1}^{0} g(x^4) \cdot (x^4)'\, dx \\
&= \int_{-1}^{0} g(\phi(x))\phi'(x)\, dx && \text{Change of Variable } \phi(x) = x^4 \\
&= \int_{\phi(-1)}^{\phi(0)} g(t)\, dt \\
&= \int_{1}^{0} g(t)\, dt \\
&= -\int_{0}^{1} g(t)\, dt \\
&= -\int_{0}^{1} g(x)\, dx
\end{aligned}$$

Thus, $\int_{-1}^{0} f(x)\, dx + \int_0^1 g(x)\, dx = 0$.

8. **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty} \frac{1}{a^k}\left[1 - \frac{1}{(k+1)a} + \frac{1}{k}\right] = \frac{1}{132}.$$

Hint: Use Telescoping and Geometric Series.

Solution. We consider

$$\frac{1}{a^k}\left[1 - \frac{1}{(k+1)a} + \frac{1}{k}\right] = \frac{1}{a^k} - \frac{1}{(k+1)a^{k+1}} + \frac{1}{ka^k} = \frac{1}{a^k} + \left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right).$$

So, the above sequence consist of a geometric and telescoping sequences. It follows that

$$\begin{aligned}
\frac{1}{132} = \sum_{k=1}^{\infty} \frac{1}{a^k}\left[1 - \frac{1}{(k+1)a} + \frac{1}{k}\right] &= \sum_{k=1}^{\infty}\left[\frac{1}{a^k} + \left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right)\right] \\
&= \sum_{k=1}^{\infty} \frac{1}{a^k} + \sum_{k=1}^{\infty}\left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right). \\
&= \frac{\frac{1}{a}}{1-\frac{1}{a}} + \left(\frac{1}{a} - \lim_{k\to\infty} \frac{1}{(k+1)a^{k+1}}\right) \qquad \text{if } |a| > 1 \\
&= \frac{1}{a-1} + \left(\frac{1}{a} - 0\right) = \frac{2a-1}{a(a-1)}
\end{aligned}$$

We obtain $a(a-1) = 132(2a-1)$ or $a^2 - 265a + 132 = 0$. Then,

$$a = \frac{265 - \sqrt{265^2 - 4(1)(132)}}{2(1)} < 1 \quad \text{and} \quad a = \frac{265 + \sqrt{265^2 - 4(1)(132)}}{2(1)} > 1$$

Thus, $a = \frac{265+\sqrt{265^2-4(1)(132)}}{2(1)}$ #

EDIT: Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty} \frac{1}{a^k}\left[1 + \frac{1}{(k+1)a} - \frac{1}{k}\right] = \frac{1}{132}.$$

Hint: Use Telescoping and Geometric Series.

Solution. We consider

$$\frac{1}{a^k}\left[1 + \frac{1}{(k+1)a} - \frac{1}{k}\right] = \frac{1}{a^k} + \frac{1}{(k+1)a^{k+1}} - \frac{1}{ka^k} = \frac{1}{a^k} - \left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right).$$

So, the above sequence consist of a geometric and telescoping sequences. It follows that

$$\begin{aligned}
\frac{1}{132} = \sum_{k=1}^{\infty} \frac{1}{a^k}\left[1 + \frac{1}{(k+1)a} - \frac{1}{k}\right] &= \sum_{k=1}^{\infty}\left[\frac{1}{a^k} - \left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right)\right] \\
&= \sum_{k=1}^{\infty} \frac{1}{a^k} - \sum_{k=1}^{\infty}\left(\frac{1}{ka^k} - \frac{1}{(k+1)a^{k+1}}\right). \\
&= \frac{\frac{1}{a}}{1-\frac{1}{a}} - \left(\frac{1}{a} - \lim_{k\to\infty} \frac{1}{(k+1)a^{k+1}}\right) \qquad \text{if } |a| > 1 \\
&= \frac{1}{a-1} - \left(\frac{1}{a} - 0\right) = \frac{1}{a(a-1)}
\end{aligned}$$

We obtain $132 = a(a-1)$ or $(a-12)(a+11) = a^2 - a - 132 = 0$. Thus, $a = 12, -11$ #

9. **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \left(a + (-1)^k\right)^k \text{ converges or NOT.}$$

Verify your answer.

Solution. Claim that the series diverges.

Proof. Assume that $\sum_{k=1}^{\infty} \left(a + (-1)^k\right)^k$ converges. By the Root Test, $r < 1$ if

$$\begin{aligned} r &= \limsup_{k\to\infty} \left| \left(a + (-1)^k\right)^k \right|^{\frac{1}{k}} \\ &= \limsup_{k\to\infty} \left| a + (-1)^k \right| \\ &= \lim_{n\to\infty} \sup\{|a-1|, |a+1|\} \\ &= \sup\{|a-1|, |a+1|\} \end{aligned}$$

If $a = 0$, then $r = \sup\{1\} = 1$. This contradicts $r < 1$.
Suppose that $a \neq 0$.
Case $r = |a-1| < 1$. Then

$$|a+1| < \sup\{|a-1|, |a+1|\} = |a-1| \qquad (*)$$

We obatian

$$\begin{array}{rcccl} -1 & < & a-1 & < & 1 \\ 0 & < & a & < & 2 \\ 1 & < & a+1 & < & 3 \end{array}$$

So, $|a+1| > 1$. We get $|a+1| > 1 > |a-1|$. It contradicts $(*)$.
Case $r = |a+1| < 1$. Then

$$|a-1| < \sup\{|a-1|, |a+1|\} = |a+1| \qquad (**)$$

We obatian

$$\begin{array}{rcccl} -1 & < & a+1 & < & 1 \\ -2 & < & a & < & 0 \\ -3 & < & a-1 & < & -1 \end{array}$$

So, $|a-1| > 1$. We get $|a-1| > 1 > |a+1|$. It contradicts $(**)$.
Therefore, $\sum_{k=1}^{\infty} \left(a + (-1)^k\right)^k$ diverges. □

10. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{k+1}{k^2+1}\right)$$

is conditionally convergent.

Solution. Firstly, we see that

$$\lim_{k\to\infty}\frac{k+1}{k^2+1}=\lim_{k\to\infty}\frac{k(1+\frac{1}{k})}{k^2(1+\frac{1}{k^2})}=\lim_{k\to\infty}\frac{1}{k}\left(\frac{1+\frac{1}{k}}{1+\frac{1}{k^2}}\right)=0\cdot 1=0.$$

Next, let $f(x)=\dfrac{x+1}{x^2+1}$ where $x\geq 1$. The derivative of $f(x)$ is

$$\begin{aligned}f'(x)&=\frac{(x^2+1)\cdot 1-(x+1)\cdot 2x}{(x^2+1)^2}\\&=\frac{x^2+1-2x^2-2x}{(x^2+1)^2}\\&=\frac{1-2x-x^2}{(x^2+1)^2}=\frac{2-(1+2x+x^2)}{(x^2+1)^2}\\&=\frac{2-(x+1)^2}{(x^2+1)^2}<0\quad\text{for all } x\geq 1.\end{aligned}$$

So, $\left\{\dfrac{k+1}{k^2+1}\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{k+1}{k^2+1}\right)\qquad\text{converges.}$$

Finally, we consider

$$\sum_{k=1}^{\infty}\left|(-1)^k\left(\frac{k+1}{k^2+1}\right)\right|=\sum_{k=1}^{\infty}\left(\frac{k+1}{k^2+1}\right)$$

and

$$\lim_{k\to\infty}\frac{\left(\frac{k+1}{k^2+1}\right)}{\frac{1}{k}}=\lim_{k\to\infty}\frac{k(k+1)}{k^2+1}=\lim_{k\to\infty}\left(\frac{k^2+k}{k^2+1}\right)=1>0$$

Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges ($p=1$), by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty}\left(\frac{k+1}{k^2+1}\right)\qquad\text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{k+1}{k^2+1}\right)\qquad\text{is conditionally convergent.}$$