Suan Sunandha Rajabhat University
Faculty of Education
Division of Mathematics
Final Examination Semester 2/2021

Subject	Mathematical Analysis
ID	MAC3310
Place	Zoom
Time	1 p.m. (3 hours 30 minutes) Wendsday 23 March 2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.
Marks	100 (30%)

No.1

1. **(10 marks)** Use definition to prove that
$$f(x) = \frac{x}{x^2+1}$$
is continuous at $x = 1$.

2. **(10 marks)** Use definition to prove that
$$f(x) = \frac{x}{x^2+1}$$
is continuous at $x = -1$.

3. **(10 marks)** Use definition to prove that
$$f(x) = \frac{x}{x^2+2}$$
is continuous at $x = 1$.

4. **(10 marks)** Use definition to prove that
$$f(x) = \frac{x}{x^2+2}$$
is continuous at $x = -1$.

No.2

1. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define
$$g(x) = x^2 + f(x) \quad \text{where } x \in [0,1].$$
Prove that g is uniformly continuous on $[0,1]$.

2. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define
$$g(x) = 2x^2 + f(x) \quad \text{where } x \in [0,1].$$
Prove that g is uniformly continuous on $[0,1]$.

3. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define
$$g(x) = x^2 - f(x) \quad \text{where } x \in [0,1].$$
Prove that g is uniformly continuous on $[0,1]$.

4. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define
$$g(x) = 2x^2 - f(x) \quad \text{where } x \in [0,1].$$
Prove that g is uniformly continuous on $[0,1]$.

No.3

1. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 1 \leq \frac{x^2+1}{2} \quad \text{for all} \quad x \geq 1.$$

2. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 2 \leq \frac{x^2+3}{2} \quad \text{for all} \quad x \geq 1.$$

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 3 \leq \frac{x^2+5}{2} \quad \text{for all} \quad x \geq 1.$$

4. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 4 \leq \frac{x^2+7}{2} \quad \text{for all} \quad x \geq 1.$$

No.4

1. **(10 marks)** Let $f(x) = x + e^x$ where $x \in \mathbb{R}$.

   1.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   1.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

   1.3 **(3 marks)** Compute $(f^{-1})'(2 + \ln 2)$.

2. **(10 marks)** Let $f(x) = 2x + e^x$ where $x \in \mathbb{R}$.

   2.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   2.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

   2.3 **(3 marks)** Compute $(f^{-1})'(2 + 2\ln 2)$.

3. **(10 marks)** Let $f(x) = x + e^{2x}$ where $x \in \mathbb{R}$.

   3.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   3.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

   3.3 **(3 marks)** Compute $(f^{-1})'(4 + \ln 2)$.

4. **(10 marks)** Let $f(x) = 2x + e^{2x}$ where $x \in \mathbb{R}$.

   4.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   4.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

   4.3 **(3 marks)** Compute $(f^{-1})'(4 + 2\ln 2)$.

No.5

1. **(10 marks)** Define
$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 2 \\ 1 & \text{if } x \in (0, 2) \end{cases}$$
Use definition to show that f is integrable on $[0, 2]$

2. **(10 marks)** Define
$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 3 \\ 1 & \text{if } x \in (0, 3) \end{cases}$$
Use definition to show that f is integrable on $[0, 3]$

3. **(10 marks)** Define
$$f(x) = \begin{cases} 0 & \text{if } x = 0, 2, 3 \\ 1 & \text{if } x \in (0, 3) \end{cases}$$
Use definition to show that f is integrable on $[0, 3]$

4. **(10 marks)** Define
$$f(x) = \begin{cases} 0 & \text{if } x = 1, 2, 3 \\ 1 & \text{if } x \in (1, 3) \end{cases}$$
Use definition to show that f is integrable on $[1, 3]$

No.6

1. **(10 marks)** Let $f(x) = \sqrt{x}$ where $x \in [0,1]$ and $P = \left\{ \frac{j^2}{n^2} : j = 0, 1, ..., n \right\}$ be a partition of $[0,1]$.

   1.1 **(4 marks)** Let $x_j = \frac{j^2}{n^2}$ for each $j = 0, 1, ..., n$. Find Δx_j and show that $\|P\| \to 0$ as $n \to \infty$.

   1.2 **(6 marks)** If the Riemann sum converges to $I(f)$, what is $I(f)$.

2. **(10 marks)** Let $f(x) = \sqrt{x}$ where $x \in [0,1]$ and $P = \left\{ \frac{j^2}{n^4} : j = 0, 1, ..., n^2 \right\}$ be a partition of $[0,1]$.

   2.1 **(4 marks)** Let $x_j = \frac{j^2}{n^4}$ for each $j = 0, 1, ..., n^2$. Find Δx_j and show that $\|P\| \to 0$ as $n \to \infty$.

   2.2 **(6 marks)** If the Riemann sum converges to $I(f)$, what is $I(f)$.

No.7

1. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} g(t) \cdot \sqrt{t}\, dt.$$

   Show that $\int_0^1 xg(x) + f(x)\, dx = 0$.

   Hint: Use integration by part to $\int_0^1 xf'(x)\, dx$.

2. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^4} g(t) \cdot \sqrt{t}\, dt.$$

   Show that $\int_0^1 xg(x) + 2xf(x)\, dx = 0$.

   Hint: Use integration by part to $\int_0^1 x^2 f'(x)\, dx$.

No.8

1. **(10 marks)** Let π be a Pi constant. Show that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right]$$

   converges and find its value.
   Hint: Use Telescoping Series.

2. **(10 marks)** Let π be a Pi constant. Show that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \left(\frac{\pi^k}{\pi} \right)^4 \right]$$

   converges and find its value.
   Hint: Use Telescoping Series.

No.9

1. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

   Hint: Use Cauchy criterion

2. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges absolutely and } \sum_{k=1}^{\infty} b_k \text{ converges, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

   Hint: Use Cauchy criterion

3. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges absolutely.}$$

   Hint: Use Cauchy criterion

4. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges absolutely and } \sum_{k=1}^{\infty} b_k \text{ converges, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges absolutely.}$$

   Hint: Use Cauchy criterion

No.10

1. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty} (-1)^k \arcsin\left(\frac{1}{k}\right)$$

   is conditionally convergent.

2. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty} (-1)^k \sin\left(\frac{1}{k}\right)$$

   is conditionally convergent.

3. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty} (-1)^k \tan\left(\frac{1}{k}\right)$$

   is conditionally convergent.

Solution Final: MAC3309 Mathematical Analysis

No.1

1. **(10 marks)** Use definition to prove that $f(x) = \dfrac{x}{x^2+1}$ is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \sqrt{2\varepsilon}$ such that $|x-1| < \delta$. Then

$$|x-1|^2 < \delta^2 = 2\varepsilon.$$

By the fact that $x^2+1 \geq 1$ for all $x \in \mathbb{R}$, we obtain

$$\frac{1}{x^2+1} \leq 1.$$

From two reasons, it leads to the below inequality:

$$\begin{aligned}
|f(x) - f(1)| &= \left| \frac{x}{x^2+1} - \frac{1}{2} \right| = \left| \frac{2x - (x^2+1)}{2(x^2+1)} \right| \\
&= \left| \frac{-(x^2-2x+1)}{2(x^2+1)} \right| = \left| \frac{(x-1)^2}{2(x^2+1)} \right| \\
&= \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot |x-1|^2 < \frac{1}{2} \cdot 1 \cdot \delta^2 = \frac{1}{2} \cdot 2\varepsilon = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at $x = 1$. □

2. **(10 marks)** Use definition to prove that $f(x) = \dfrac{x}{x^2+1}$ is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \sqrt{2\varepsilon}$ such that $|x+1| < \delta$. Then

$$|x+1|^2 < \delta^2 = 2\varepsilon.$$

By the fact that $x^2+1 \geq 1$ for all $x \in \mathbb{R}$, we obtain

$$\frac{1}{x^2+1} \leq 1.$$

From two reasons, it leads to the below inequality:

$$\begin{aligned}
|f(x) - f(-1)| &= \left| \frac{x}{x^2+1} + \frac{1}{2} \right| = \left| \frac{2x + (x^2+1)}{2(x^2+1)} \right| \\
&= \left| \frac{x^2+2x+1}{2(x^2+1)} \right| = \left| \frac{(x+1)^2}{2(x^2+1)} \right| \\
&= \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot |x+1|^2 < \frac{1}{2} \cdot 1 \cdot \delta^2 = \frac{1}{2} \cdot 2\varepsilon = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at $x = -1$. □

3. **(10 marks)** Use definition to prove that $f(x) = \dfrac{x}{x^2+2}$ is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{2}\}$ such that $|x-1| < \delta$. Then $|x-1| < 1$. We obtain

$$|x| - 1 < |x-1| < 1. \text{ So, } |x| < 2.$$

By the fact that $x^2 + 1 \geq 1$ for all $x \in \mathbb{R}$, we obtain

$$\frac{1}{x^2+1} \leq 1.$$

From three reasons, it leads to the below inequality:

$$\begin{aligned}
|f(x) - f(1)| &= \left|\frac{x}{x^2+2} - \frac{1}{3}\right| = \left|\frac{3x - (x^2+2)}{2(x^2+2)}\right| \\
&= \left|\frac{-(x^2-3x+2)}{2(x^2+1)}\right| = \left|\frac{(x-1)(x-2)}{2(x^2+1)}\right| \\
&= \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot |x-2| \cdot |x-1| \\
&< \frac{1}{2} \cdot 1 \cdot (|x|+2) \cdot \delta = \frac{1}{2} \cdot 1 \cdot (2+2) \cdot \delta \\
&= 2\delta < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at $x = 1$. □

4. **(10 marks)** Use definition to prove that $f(x) = \dfrac{x}{x^2+2}$ is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{2}\}$ such that $|x+1| < \delta$. Then $|x+1| < 1$. We obtain

$$|x| - 1 < |x+1| < 1. \text{ So, } |x| < 2.$$

By the fact that $x^2 + 1 \geq 1$ for all $x \in \mathbb{R}$, we obtain

$$\frac{1}{x^2+1} \leq 1.$$

From three reasons, it leads to the below inequality:

$$\begin{aligned}
|f(x) - f(-1)| &= \left|\frac{x}{x^2+2} + \frac{1}{3}\right| = \left|\frac{3x + (x^2+2)}{2(x^2+2)}\right| \\
&= \left|\frac{x^2+3x+2}{2(x^2+1)}\right| = \left|\frac{(x+1)(x+2)}{2(x^2+1)}\right| \\
&= \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot |x+2| \cdot |x+1| \\
&< \frac{1}{2} \cdot 1 \cdot (|x|+2) \cdot \delta = \frac{1}{2} \cdot 1 \cdot (2+2) \cdot \delta \\
&= 2\delta < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at $x = -1$. □

No.2

1. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define

$$g(x) = x^2 + f(x) \quad \text{where } x \in [0,1].$$

Prove that g is uniformly continuous on $[0,1]$.

Proof. Assume that f be uniformly continuous on I.
Let $\varepsilon > 0$. There is an $\delta_1 > 0$ such that

$$|x-a| < \delta_1 \text{ for all } x, a \in [0,1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_1, \frac{\varepsilon}{4}\right\}$. Let $x, a \in [0,1]$ such that $|x-a| < \delta$. Then $0 \le x + a \le 2$ and $|x - a| < \delta_1$.
We obatin

$$\begin{aligned} |g(x) - g(a)| &= |x^2 + f(x) - a^2 - f(a)| \\ &= |(x-a)(x+a) + f(x) - f(a)| \\ &\le |x-a||x+a| + |f(x) - f(a)| \\ &< \delta \cdot 2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{4} \cdot 2 + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, g is uniformly continuous on $[0,1]$. □

2. **(10 marks)** Let $f : [0,1] \to \mathbb{R}$ be uniformly continuous on $[0,1]$. Define

$$g(x) = 2x^2 + f(x) \quad \text{where } x \in [0,1].$$

Prove that g is uniformly continuous on $[0,1]$.

Proof. Assume that f be uniformly continuous on I.
Let $\varepsilon > 0$. There is an $\delta_1 > 0$ such that

$$|x-a| < \delta_1 \text{ for all } x, a \in [0,1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_1, \frac{\varepsilon}{8}\right\}$. Let $x, a \in [0,1]$ such that $|x-a| < \delta$. Then $0 \le x + a \le 2$ and $|x - a| < \delta_1$.
We obatin

$$\begin{aligned} |g(x) - g(a)| &= |2x^2 + f(x) - 2a^2 - f(a)| \\ &= |2(x-a)(x+a) + f(x) - f(a)| \\ &\le 2|x-a||x+a| + |f(x) - f(a)| \\ &< 2\delta \cdot 2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{8} \cdot 4 + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, g is uniformly continuous on $[0,1]$. □

3. **(10 marks)** Let $f : [0, 1] \to \mathbb{R}$ be uniformly continuous on $[0, 1]$. Define

$$g(x) = x^2 - f(x) \quad \text{where } x \in [0, 1].$$

Prove that g is uniformly continuous on $[0, 1]$.

Proof. Assume that f be uniformly continuous on I.
Let $\varepsilon > 0$. There is an $\delta_1 > 0$ such that

$$|x - a| < \delta_1 \text{ for all } x, a \in [0, 1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_1, \frac{\varepsilon}{4}\right\}$. Let $x, a \in [0, 1]$ such that $|x - a| < \delta$. Then $0 \le x + a \le 2$ and $|x - a| < \delta_1$. We obatin

$$\begin{aligned} |g(x) - g(a)| &= |x^2 - f(x) - a^2 + f(a)| \\ &= |(x - a)(x + a) - (f(x) - f(a))| \\ &\le |x - a||x + a| + |f(x) - f(a)| \\ &< \delta \cdot 2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{4} \cdot 2 + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, g is uniformly continuous on $[0, 1]$. □

4. **(10 marks)** Let $f : [0, 1] \to \mathbb{R}$ be uniformly continuous on $[0, 1]$. Define

$$g(x) = 2x^2 - f(x) \quad \text{where } x \in [0, 1].$$

Prove that g is uniformly continuous on $[0, 1]$.

Proof. Assume that f be uniformly continuous on I.
Let $\varepsilon > 0$. There is an $\delta_1 > 0$ such that

$$|x - a| < \delta_1 \text{ for all } x, a \in [0, 1] \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_1, \frac{\varepsilon}{8}\right\}$. Let $x, a \in [0, 1]$ such that $|x - a| < \delta$. Then $0 \le x + a \le 2$ and $|x - a| < \delta_1$. We obatin

$$\begin{aligned} |g(x) - g(a)| &= |2x^2 - f(x) - 2a^2 + f(a)| \\ &= |2(x - a)(x + a) - (f(x) - f(a))| \\ &\le 2|x - a||x + a| + |f(x) - f(a)| \\ &< 2\delta \cdot 2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{8} \cdot 4 + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, g is uniformly continuous on $[0, 1]$. □

No.3

1. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 1 \le \frac{x^2+1}{2} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \ln x + 1 - \frac{x^2+1}{2} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = \frac{1}{x} - x$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\ln a + 1 - \frac{a^2+1}{2} = \left(\frac{1}{c} - c\right)(a-1) = \left(\frac{1-c^2}{c}\right)(a-1)$$

From $1 < c$, it leads to $1 - c^2 < 0$. So,

$$\frac{1-c^2}{c} < 0.$$

Since $a > 1$, $a - 1 > 0$. Therefore,

$$\ln a + 1 - \frac{a^2+1}{2} = \left(\frac{1-c^2}{c}\right)(a-1) < 0$$

Therefore, We conclude that $\ln x + 1 \le \frac{x^2+1}{2}$ for all $x \ge 1$. □

2. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 2 \le \frac{x^2+3}{2} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \ln x + 2 - \frac{x^2+3}{2} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = \frac{1}{x} - x$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\ln a + 2 - \frac{a^2+3}{2} = \left(\frac{1}{c} - c\right)(a-1) = \left(\frac{1-c^2}{c}\right)(a-1)$$

From $1 < c$, it leads to $1 - c^2 < 0$. So,

$$\frac{1-c^2}{c} < 0.$$

Since $a > 1$, $a - 1 > 0$. Therefore,

$$\ln a + 2 - \frac{a^2+3}{2} = \left(\frac{1-c^2}{c}\right)(a-1) < 0$$

Therefore, We conclude that $\ln x + 2 \le \frac{x^2+3}{2}$ for all $x \ge 1$. □

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 3 \le \frac{x^2+5}{2} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \ln x + 3 - \frac{x^2+5}{2} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = \frac{1}{x} - x$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\ln a + 3 - \frac{a^2+5}{2} = \left(\frac{1}{c} - c\right)(a-1) = \left(\frac{1-c^2}{c}\right)(a-1)$$

From $1 < c$, it leads to $1 - c^2 < 0$. So,

$$\frac{1-c^2}{c} < 0.$$

Since $a > 1$, $a - 1 > 0$. Therefore,

$$\ln a + 3 - \frac{a^2+5}{2} = \left(\frac{1-c^2}{c}\right)(a-1) < 0$$

Therefore, We conclude that $\ln x + 3 \le \frac{x^2+5}{2}$ for all $x \ge 1$. □

4. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\ln x + 4 \le \frac{x^2+7}{2} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \ln x + 4 - \frac{x^2+7}{2} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = \frac{1}{x} - x$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\ln a + 4 - \frac{a^2+7}{2} = \left(\frac{1}{c} - c\right)(a-1) = \left(\frac{1-c^2}{c}\right)(a-1)$$

From $1 < c$, it leads to $1 - c^2 < 0$. So,

$$\frac{1-c^2}{c} < 0.$$

Since $a > 1$, $a - 1 > 0$. Therefore,

$$\ln a + 4 - \frac{a^2+7}{2} = \left(\frac{1-c^2}{c}\right)(a-1) < 0$$

Therefore, We conclude that $\ln x + 4 \le \frac{x^2+7}{2}$ for all $x \ge 1$. □

No.4

1. **(10 marks)** Let $f(x) = x + e^x$ where $x \in \mathbb{R}$.

   1.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $x - y > 0$ and $e^x > e^y$. We obtain

$$\begin{aligned} e^y - e^x &< 0 < x - y \\ y + e^y &< x + e^x \\ f(y) &< f(x) \end{aligned}$$

   So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

   1.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

   Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

   1.3 **(3 marks)** Compute $(f^{-1})'(2 + \ln 2)$.

   Solution. We see that $f'(x) = 1 + e^x$ and $f(\ln 2) = \ln 2 + 2$. So $f^{-1}(2 + \ln 2) = \ln 2$. By IFT,

$$(f^{-1})'(2 + \ln 2) = \frac{1}{f'(f^{-1}(2 + \ln 2))} = \frac{1}{f'(\ln 2)} = \frac{1}{1+2} = \frac{1}{3} \quad \#$$

2. **(10 marks)** Let $f(x) = 2x + e^x$ where $x \in \mathbb{R}$.

   2.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

   Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $2(x - y) > 0$ and $e^x > e^y$. We obtain

$$\begin{aligned} e^y - e^x &< 0 < 2(x - y) \\ 2y + e^y &< 2x + e^x \\ f(y) &< f(x) \end{aligned}$$

   So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

   2.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

   Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

   2.3 **(3 marks)** Compute $(f^{-1})'(2 + 2\ln 2)$.

   Solution. We see that $f'(x) = 2 + e^x$ and $f(\ln 2) = 2\ln 2 + 2$. So $f^{-1}(2 + 2\ln 2) = \ln 2$. By IFT,

$$(f^{-1})'(2 + 2\ln 2) = \frac{1}{f'(f^{-1}(2 + 2\ln 2))} = \frac{1}{f'(\ln 2)} = \frac{1}{2+2} = \frac{1}{4} \quad \#$$

3. **(10 marks)** Let $f(x) = x + e^{2x}$ where $x \in \mathbb{R}$.

3.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $x - y > 0$ and $2x > 2y$. So, $e^{2x} > e^{2y}$. We obtain

$$\begin{aligned} e^{2y} - e^{2x} &< 0 < x - y \\ y + e^{2y} &< x + e^{2x} \\ f(y) &< f(x) \end{aligned}$$

So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

3.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

3.3 **(3 marks)** Compute $(f^{-1})'(4 + \ln 2)$.

Solution. We see that $f'(x) = 1 + e^{2x}$ and $f(\ln 2) = \ln 2 + 4$. So $f^{-1}(4 + \ln 2) = \ln 2$. By IFT,

$$(f^{-1})'(4 + \ln 2) = \frac{1}{f'(f^{-1}(4 + \ln 2))} = \frac{1}{f'(\ln 2)} = \frac{1}{1 + 4} = \frac{1}{5} \quad \#$$

4. **(10 marks)** Let $f(x) = 2x + e^{2x}$ where $x \in \mathbb{R}$.

4.1 **(5 marks)** Show that f^{-1} is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $2(x - y) > 0$ and $e^{2x} > e^{2y}$. We obtain

$$\begin{aligned} e^{2y} - e^{2x} &< 0 < 2(x - y) \\ 2y + e^{2y} &< 2x + e^{2x} \\ f(y) &< f(x) \end{aligned}$$

So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

4.2 **(2 marks)** Use the result from 1.1 and the Inverse Function Theorem (IFT) to explain that f differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

4.3 **(3 marks)** Compute $(f^{-1})'(4 + 2\ln 2)$.

Solution. We see that $f'(x) = 2 + 2e^{x}$ and $f(\ln 2) = 2\ln 2 + 4$. So $f^{-1}(4 + 2\ln 2) = \ln 2$. By IFT,

$$(f^{-1})'(4 + 2\ln 2) = \frac{1}{f'(f^{-1}(4 + 2\ln 2))} = \frac{1}{f'(\ln 2)} = \frac{1}{2 + 4} = \frac{1}{6} \quad \#$$

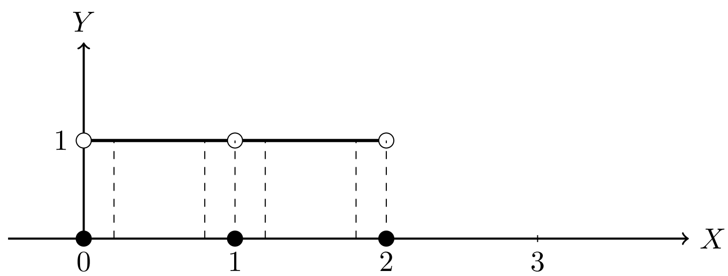
No.5

1. **(10 marks)** Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 2 \\ 1 & \text{if } x \in (0, 2) \end{cases}$$

Use definition to show that f is integrable on $[0, 2]$

Solution. A graph of the function is



Proof. Let $\varepsilon > 0$ and $k, n \in \mathbb{N}$ with $1 \leq k < n$. Choose $P = \{x_0, x_1, x_2, ..., x_k, ..., x_n\}$ where $x_0 = 0, x_k = 1$ and $x_n = 2$ by $\|P\| = \max\{\Delta x_i : i = 1, 2, ..., n\} < \frac{\varepsilon}{3}$. We obtain

$$m_j(f) = \begin{cases} 0 & \text{if} \quad j = 1, k, k+1, n \\ 1 & \text{if} \quad j = 2, 3, ..., k-1, k+2, ..., n-1 \end{cases}$$

$$M_j(f) = 1 \quad \text{if } j = 1, 2, ..., n$$

It follows that

$$\begin{aligned} L(P, f) &= \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \\ &= 0 + \sum_{j=2}^{k-1} m_j(f)(x_j - x_{j-1}) + 0 + 0 + \sum_{j=k+2}^{n-1} m_j(f)(x_j - x_{j-1}) + 0 \\ &= \sum_{j=2}^{k-1} 1(x_j - x_{j-1}) + \sum_{j=k+2}^{n-1} 1(x_j - x_{j-1}) \\ &= (x_{k-1} - x_0) + (x_{n-1} - x_{k+1}) \end{aligned}$$

$$U(P, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = x_n - x_0$$

Then

$$\begin{aligned} U(P, f) - L(P, f) &= (x_n - x_0) - [(x_{k-1} - x_0) + (x_{n-1} - x_{k+1})] \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_{k-1}) \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_k) + (x_k - x_{k-1}) \\ &= \Delta x_n + \Delta x_{k+1} + \Delta x_k \leq 3\|P\| < \varepsilon \end{aligned}$$

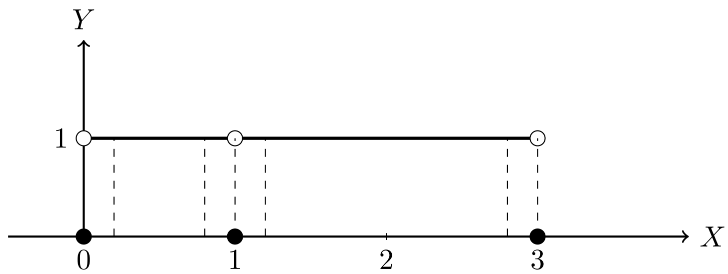
Hence, f is integrable on $[0, 2]$. □

2. **(10 marks)** Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 3 \\ 1 & \text{if } x \in (0, 3) \end{cases}$$

Use definition to show that f is integrable on $[0, 3]$

Solution. A graph of the function is



Proof. Let $\varepsilon > 0$ and $k, n \in \mathbb{N}$ with $1 \leq k < n$. Choose $P = \{x_0, x_1, x_2, ..., x_k, ..., x_n\}$ where $x_0 = 0, x_k = 1$ and $x_n = 3$ by $\|P\| = \max\{\Delta x_i : i = 1, 2, ..., n\} < \frac{\varepsilon}{3}$. We obtain

$$m_j(f) = \begin{cases} 0 & \text{if} \quad j = 1, k, k+1, n \\ 1 & \text{if} \quad j = 2, 3, ..., k-1, k+2, ..., n-1 \end{cases}$$

$$M_j(f) = 1 \quad \text{if } j = 1, 2, ..., n$$

It follows that

$$\begin{aligned} L(P, f) &= \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \\ &= 0 + \sum_{j=2}^{k-1} m_j(f)(x_j - x_{j-1}) + 0 + 0 + \sum_{j=k+2}^{n-1} m_j(f)(x_j - x_{j-1}) + 0 \\ &= \sum_{j=2}^{k-1} 1(x_j - x_{j-1}) + \sum_{j=k+2}^{n-1} 1(x_j - x_{j-1}) \\ &= (x_{k-1} - x_0) + (x_{n-1} - x_{k+1}) \end{aligned}$$

$$U(P, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = x_n - x_0$$

Then

$$\begin{aligned} U(P, f) - L(P, f) &= (x_n - x_0) - [(x_{k-1} - x_0) + (x_{n-1} - x_{k+1})] \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_{k-1}) \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_k) + (x_k - x_{k-1}) \\ &= \Delta x_n + \Delta x_{k+1} + \Delta x_k \leq 3\|P\| < \varepsilon \end{aligned}$$

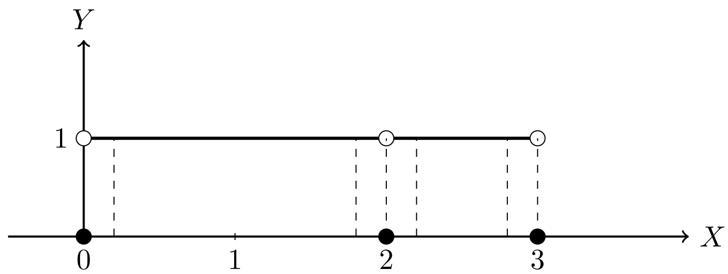
Hence, f is integrable on $[0, 3]$. $\square$

3. **(10 marks)** Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 2, 3 \\ 1 & \text{if } x \in (0, 3) \end{cases}$$

Use definition to show that f is integrable on $[0, 3]$

Solution. A graph of the function is



Proof. Let $\varepsilon > 0$ and $k, n \in \mathbb{N}$ with $1 \leq k < n$. Choose $P = \{x_0, x_1, x_2, ..., x_k, ..., x_n\}$ where $x_0 = 0, x_k = 2$ and $x_n = 3$ by $\|P\| = \max\{\Delta x_i : i = 1, 2, ..., n\} < \frac{\varepsilon}{3}$. We obtain

$$m_j(f) = \begin{cases} 0 & \text{if} \quad j = 1, k, k+1, n \\ 1 & \text{if} \quad j = 2, 3, ..., k-1, k+2, ..., n-1 \end{cases}$$

$$M_j(f) = 1 \quad \text{if } j = 1, 2, ..., n$$

It follows that

$$\begin{aligned} L(P, f) &= \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \\ &= 0 + \sum_{j=2}^{k-1} m_j(f)(x_j - x_{j-1}) + 0 + 0 + \sum_{j=k+2}^{n-1} m_j(f)(x_j - x_{j-1}) + 0 \\ &= \sum_{j=2}^{k-1} 1(x_j - x_{j-1}) + \sum_{j=k+2}^{n-1} 1(x_j - x_{j-1}) \\ &= (x_{k-1} - x_0) + (x_{n-1} - x_{k+1}) \end{aligned}$$

$$U(P, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = x_n - x_0$$

Then

$$\begin{aligned} U(P, f) - L(P, f) &= (x_n - x_0) - [(x_{k-1} - x_0) + (x_{n-1} - x_{k+1})] \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_{k-1}) \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_k) + (x_k - x_{k-1}) \\ &= \Delta x_n + \Delta x_{k+1} + \Delta x_k \leq 3\|P\| < \varepsilon \end{aligned}$$

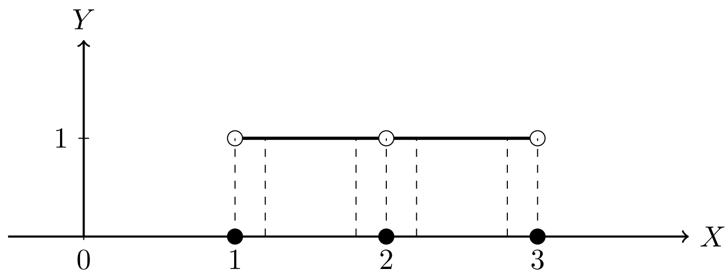
Hence, f is integrable on $[0, 3]$. $\square$

4. **(10 marks)** Define

$$f(x) = \begin{cases} 0 & \text{if } x = 1, 2, 3 \\ 1 & \text{if } x \in (1, 3) \end{cases}$$

Use definition to show that f is integrable on $[1, 3]$

Solution. A graph of the function is



Proof. Let $\varepsilon > 0$ and $k, n \in \mathbb{N}$ with $1 \leq k < n$. Choose $P = \{x_0, x_1, x_2, ..., x_k, ..., x_n\}$ where $x_0 = 1, x_k = 2$ and $x_n = 3$ by $\|P\| = \max\{\Delta x_i : i = 1, 2, ..., n\} < \frac{\varepsilon}{3}$. We obtain

$$m_j(f) = \begin{cases} 0 & \text{if} \quad j = 1, k, k+1, n \\ 1 & \text{if} \quad j = 2, 3, ..., k-1, k+2, ..., n-1 \end{cases}$$
$$M_j(f) = 1 \quad \text{if } j = 1, 2, ..., n$$

It follows that

$$\begin{aligned} L(P, f) &= \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \\ &= 0 + \sum_{j=2}^{k-1} m_j(f)(x_j - x_{j-1}) + 0 + 0 + \sum_{j=k+2}^{n-1} m_j(f)(x_j - x_{j-1}) + 0 \\ &= \sum_{j=2}^{k-1} 1(x_j - x_{j-1}) + \sum_{j=k+2}^{n-1} 1(x_j - x_{j-1}) \\ &= (x_{k-1} - x_0) + (x_{n-1} - x_{k+1}) \end{aligned}$$

$$U(P, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = x_n - x_0$$

Then

$$\begin{aligned} U(P, f) - L(P, f) &= (x_n - x_0) - [(x_{k-1} - x_0) + (x_{n-1} - x_{k+1})] \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_{k-1}) \\ &= (x_n - x_{n-1}) + (x_{k+1} - x_k) + (x_k - x_{k-1}) \\ &= \Delta x_n + \Delta x_{k+1} + \Delta x_k \leq 3\|P\| < \varepsilon \end{aligned}$$

Hence, f is integrable on $[1, 3]$. □

No.6

1. **(10 marks)** Let $f(x) = \sqrt{x}$ where $x \in [0, 1]$ and $P = \left\{ \frac{j^2}{n^2} : j = 0, 1, ..., n \right\}$ be a partition of $[0, 1]$.

   1.1 **(4 marks)** Let $x_j = \frac{j^2}{n^2}$ for each $j = 0, 1, ..., n$. Find Δx_j and show that $\|P\| \to 0$ as $n \to \infty$.
   Solution. We obtain

$$\Delta x_j = x_j - x_{j-1} = \frac{j^2}{n^2} - \frac{(j-1)^2}{n^2} = \frac{2j-1}{n^2} \quad \text{for all } j = 1, 2, 3, ..., n.$$

   We consider

$$\begin{aligned} \|P\| &= \max\{\Delta x_j : j = 1, 2, ..., n\} = \max\left\{ \frac{2j-1}{n^2} : j = 1, 2, ..., n \right\} \\ &= \max\left\{ \frac{1}{n^2}, \frac{3}{n^2}, \frac{5}{n^2}, ..., \frac{2n-1}{n^2} \right\} = \frac{2n-1}{n^2}. \end{aligned}$$

   Thus,

$$\lim_{n \to \infty} \|P\| = \lim_{n \to \infty} \frac{2n-1}{n^2} = 0.$$

   1.2 **(6 marks)** If the Riemann sum converges to $I(f)$, what is $I(f)$.
   Solution. Choose $f(t_j) = f(\frac{j^2}{n^2})$ on the subinterval $[x_{j-1}, x_j]$. We obtain

$$\begin{aligned} \sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j^2}{n^2}\right) \frac{2j-1}{n^2} = \frac{1}{n^2} \sum_{j=1}^{n} \sqrt{\frac{j^2}{n^2}} \cdot (2j-1) \\ &= \frac{1}{n^2} \sum_{j=1}^{n} \frac{j}{n} \cdot (2j-1) = \frac{1}{n^3} \sum_{j=1}^{n} (2j^2 - j) \\ &= \frac{1}{n^3} \left[2 \sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j \right] \\ &= \frac{1}{n^3} \left[2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\ &= \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \end{aligned}$$

   Thus,

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} = \frac{2}{3} - 0 = \frac{2}{3} \quad \#$$

2. **(10 marks)** Let $f(x) = \sqrt{x}$ where $x \in [0,1]$ and $P = \left\{ \frac{j^2}{n^4} : j = 0, 1, ..., n^2 \right\}$ be a partition of $[0,1]$.

2.1 **(4 marks)** Let $x_j = \frac{j^2}{n^4}$ for each $j = 0, 1, ..., n^2$. Find Δx_j and show that $\|P\| \to 0$ as $n \to \infty$.

Solution. We obtain

$$\Delta x_j = x_j - x_{j-1} = \frac{j^2}{n^4} - \frac{(j-1)^2}{n^4} = \frac{2j-1}{n^4} \qquad \text{for all } j = 1, 2, 3, ..., n^2.$$

We consider

$$\begin{aligned} \|P\| &= \max\{\Delta x_j : j = 1, 2, ..., n^2\} = \max\left\{ \frac{2j-1}{n^4} : j = 1, 2, ..., n^2 \right\} \\ &= \max\left\{ \frac{1}{n^4}, \frac{3}{n^4}, \frac{5}{n^4}, ..., \frac{2n^2-1}{n^4} \right\} = \frac{2n^2-1}{n^4}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|P\| = \lim_{n\to\infty} \frac{2n^2-1}{n^4} = 0.$$

2.2 **(6 marks)** If the Riemann sum converges to $I(f)$, what is $I(f)$.

Solution. Choose $f(t_j) = f(\frac{j^2}{n^4})$ on the subinterval $[x_{j-1}, x_j]$. We obtain

$$\begin{aligned} \sum_{j=1}^{n^2} f(t_j)\Delta x_j &= \sum_{j=1}^{n^2} f\left(\frac{j^2}{n^4}\right) \frac{2j-1}{n^4} = \frac{1}{n^4}\sum_{j=1}^{n^2} \sqrt{\frac{j^2}{n^4}} \cdot (2j-1) \\ &= \frac{1}{n^4}\sum_{j=1}^{n^2} \frac{j}{n^2} \cdot (2j-1) = \frac{1}{n^6}\sum_{j=1}^{n^2}(2j^2 - j) \\ &= \frac{1}{n^6}\left[2\sum_{j=1}^{n^2} j^2 - \sum_{j=1}^{n^2} j \right] \\ &= \frac{1}{n^6}\left[2 \cdot \frac{n^2(n^2+1)(2n^2+1)}{6} - \frac{n^2(n^2+1)}{2} \right] \\ &= \frac{(n^2+1)(2n^2+1)}{3n^4} - \frac{n^2+1}{2n^4} \end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty} \frac{(n^2+1)(2n^2+1)}{3n^4} - \frac{n^2+1}{2n^4} = \frac{2}{3} - 0 = \frac{2}{3} \quad \#$$

No.7

1. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^2} g(t) \cdot \sqrt{t}\, dt.$$

Show that $\displaystyle\int_0^1 xg(x) + f(x)\, dx = 0$.

Hint: Use integration by part to $\displaystyle\int_0^1 xf'(x)\, dx$.

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = g(x^2) \cdot \sqrt{x^2} \cdot 2x = g(x^2) \cdot 2x|x|.$$

By integration by part, we obtain

$$\begin{aligned}
\int_0^1 xf'(x)\, dx &= [xf(x)]_0^1 - \int_0^1 (x)' f(x)\, dx \\
\int_0^1 x \cdot g(x^2) \cdot 2x|x|\, dx &= f(1) - \int_0^1 f(x)\, dx \\
\int_0^1 2x^3 \cdot g(x^2)\, dx &= \int_1^1 g(t) \cdot \sqrt{t}\, dt - \int_0^1 f(x)\, dx \\
\int_0^1 x^2 \cdot g(x^2) \cdot (2x)\, dx &= 0 - \int_0^1 f(x)\, dx \\
\int_0^1 x^2 \cdot g(x^2) \cdot (x^2)'\, dx &= 0 - \int_0^1 f(x)\, dx && \text{Change of Variable } \phi(x) = x^2 \\
\int_0^1 \phi(x) \cdot g(\phi(x)) \cdot \phi'(x)\, dx &= 0 - \int_0^1 f(x)\, dx \\
\int_{\phi(0)}^{\phi(1)} t \cdot g(t)\, dt + \int_0^1 f(x)\, dx &= 0 \\
\int_0^1 xg(x)\, dx + \int_0^1 f(x)\, dx &= 0 \\
\int_0^1 xg(x) + f(x)\, dx &= 0
\end{aligned}$$

2. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{x^4} g(t) \cdot \sqrt{t}\, dt.$$

Show that $\displaystyle\int_0^1 xg(x) + 2xf(x)\, dx = 0$.

Hint: Use integration by part to $\displaystyle\int_0^1 x^2 f'(x)\, dx$.

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = g(x^4) \cdot \sqrt{x^4} \cdot 4x^3 = g(x^4) \cdot 4x^5.$$

By integration by part, we obtain

$$\int_0^1 x^2 f'(x)\, dx = [x^2 f(x)]_0^1 - \int_0^1 (x^2)' f(x)\, dx$$

$$\int_0^1 x^2 \cdot g(x^4) \cdot 4x^5\, dx = f(1) - \int_0^1 2xf(x)\, dx$$

$$\int_0^1 4x^7 \cdot g(x^4)\, dx = \int_1^1 g(t) \cdot \sqrt{t}\, dt - \int_0^1 2xf(x)\, dx$$

$$\int_0^1 x^4 \cdot g(x^4) \cdot (4x^3)\, dx = 0 - \int_0^1 2xf(x)\, dx$$

$$\int_0^1 x^4 \cdot g(x^4) \cdot (x^4)'\, dx = 0 - \int_0^1 2xf(x)\, dx \qquad \text{Change of Variable } \phi(x) = x^4$$

$$\int_0^1 \phi(x) \cdot g(\phi(x)) \cdot \phi'(x)\, dx = 0 - \int_0^1 2xf(x)\, dx$$

$$\int_{\phi(0)}^{\phi(1)} t \cdot g(t)\, dt + \int_0^1 2xf(x)\, dx = 0$$

$$\int_0^1 xg(x)\, dx + \int_0^1 2xf(x)\, dx = 0$$

$$\int_0^1 xg(x) + 2xf(x)\, dx = 0$$

No.8

1. **(10 marks)** Let π be a Pi constant. Show that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right]$$

converges and find its value.
Hint: Use Telescoping Series.

Solution. We rewrite the term of this series

$$\begin{aligned} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right] &= \frac{1}{\pi^{k^2}} - \frac{1}{\pi^{k^2-2k+1}} + \frac{1}{\pi^{k^2}} \cdot \frac{\pi^{k^2}}{\pi^k} \\ &= \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \left(\frac{1}{\pi} \right)^k \end{aligned}$$

Then, the first term is telescoping series and the second term is geometric series. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right] &= \sum_{k=1}^{\infty} \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi} \right)^k \\ &= -\sum_{k=1}^{\infty} \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{k^2}} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi} \right)^k \\ &= -1 + \lim_{k \to \infty} \frac{1}{\pi^{k^2}} + \frac{\frac{1}{\pi}}{1 - \frac{1}{\pi}} \\ &= -1 + 0 + \frac{1}{\pi - 1} = \frac{2 - \pi}{\pi - 1} \quad \# \end{aligned}$$

2. **(10 marks)** Let π be a Pi constant. Show that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \left(\frac{\pi^k}{\pi} \right)^4 \right]$$

converges and find its value.
Hint: Use Telescoping Series.

Solution. We rewrite the term of this series

$$\begin{aligned} \frac{1}{\pi^{k^2}} \left[1 - \left(\frac{\pi^k}{\pi} \right)^4 \right] &= \frac{1}{\pi^{k^2}} - \frac{1}{\pi^{k^2}} \cdot \frac{\pi^{4k}}{\pi^4} = \frac{1}{\pi^{k^2}} - \frac{1}{\pi^{k^2-4k+4}} = \frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-2)^2}} \\ &= \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{(k-2)^2}} \right) \end{aligned}$$

Then, the two terms are telescoping series. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \left(\frac{\pi^k}{\pi} \right)^4 \right] &= \sum_{k=1}^{\infty} \left[\left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{(k-2)^2}} \right) \right] \\ &= -\sum_{k=1}^{\infty} \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{k^2}} \right) - \sum_{k=1}^{\infty} \left(\frac{1}{\pi^{(k-2)^2}} - \frac{1}{\pi^{(k-1)^2}} \right) \\ &= -1 + \lim_{k \to \infty} \frac{1}{\pi^{k^2}} - \frac{1}{\pi} + \lim_{k \to \infty} \frac{1}{\pi^{(k-1)^2}} \\ &= -1 + 0 - \frac{1}{\pi} + 0 = -\frac{\pi + 1}{\pi} \quad \# \end{aligned}$$

No.9

1. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

Hint: Use Cauchy criterion

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ converges absolutely. Then $\{a_k\}$ converges (to zero). So, $\{a_k\}$ is bounded, i.e., there is an $M > 0$ such that

$$|a_k| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} b_k$ converges absolutely, $\sum_{k=1}^{\infty} |b_k|$ converges. By Cauchy criterion, there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |b_k| < \frac{\varepsilon}{M}.$$

Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. If $n \leq k \leq m$, then $\frac{1}{k} \leq 1$. We obatin

$$\begin{aligned}\sum_{k=n}^{m} |a_k b_k| &= \sum_{k=n}^{m} |a_k||b_k| \leq \sum_{k=n}^{m} M|b_k| \\ &= M \sum_{k=n}^{m} |b_k| < M \cdot \frac{\varepsilon}{M} = \varepsilon.\end{aligned}$$

Thus, $\sum_{k=1}^{\infty} |a_k b_k|$ converges. This result concluded that $\sum_{k=1}^{\infty} a_k b_k$ converges.

$\square$

2. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges absolutely and } \sum_{k=1}^{\infty} b_k \text{ converges, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

Hint: Use Cauchy criterion

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{k=1}^{\infty} b_k$ converges. Then $\{b_k\}$ converges (to zero). So, $\{b_k\}$ is bounded, i.e., there is an $M > 0$ such that

$$|b_k| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, $\sum_{k=1}^{\infty} |a_k|$ converges. By Cauchy criterion, there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \frac{\varepsilon}{M}.$$

Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. If $n \leq k \leq m$, then $\frac{1}{k} \leq 1$. We obatin

$$\begin{aligned} \sum_{k=n}^{m} |a_k b_k| &= \sum_{k=n}^{m} |a_k||b_k| \leq \sum_{k=n}^{m} M|a_k| \\ &= M \sum_{k=n}^{m} |a_k| < M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Thus, $\sum_{k=1}^{\infty} |a_k b_k|$ converges. This result concluded that $\sum_{k=1}^{\infty} a_k b_k$ converges.

□

3. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges and } \sum_{k=1}^{\infty} b_k \text{ converges absolutely, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges absolutely.}$$

Hint: Use Cauchy criterion

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ converges absolutely. Then $\{a_k\}$ converges (to zero). So, $\{a_k\}$ is bounded, i.e., there is an $M > 0$ such that

$$|a_k| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} b_k$ converges absolutely, $\sum_{k=1}^{\infty} |b_k|$ converges. By Cauchy criterion, there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |b_k| < \frac{\varepsilon}{M}.$$

Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. If $n \leq k \leq m$, then $\frac{1}{k} \leq 1$. We obatin

$$\begin{aligned} \sum_{k=n}^{m} |a_k b_k| &= \sum_{k=n}^{m} |a_k||b_k| \leq \sum_{k=n}^{m} M|b_k| \\ &= M \sum_{k=n}^{m} |b_k| < M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Thus, $\sum_{k=1}^{\infty} |a_k b_k|$ converges. On other word, we said that $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

$\square$

4. **(10 marks)** Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. Prove that

$$\text{if } \sum_{k=1}^{\infty} a_k \text{ converges absolutely and } \sum_{k=1}^{\infty} b_k \text{ converges, then } \sum_{k=1}^{\infty} a_k b_k \text{ converges absolutely.}$$

Hint: Use Cauchy criterion

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{k=1}^{\infty} b_k$ converges. Then $\{b_k\}$ converges (to zero). So, $\{b_k\}$ is bounded, i.e., there is an $M > 0$ such that

$$|b_k| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, $\sum_{k=1}^{\infty} |a_k|$ converges. By Cauchy criterion, there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \frac{\varepsilon}{M}.$$

Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. If $n \leq k \leq m$, then $\frac{1}{k} \leq 1$. We obatin

$$\begin{aligned} \sum_{k=n}^{m} |a_k b_k| &= \sum_{k=n}^{m} |a_k||b_k| \leq \sum_{k=n}^{m} M|a_k| \\ &= M \sum_{k=n}^{m} |a_k| < M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Thus, $\sum_{k=1}^{\infty} |a_k b_k|$ converges. On other word, we said that $\sum_{k=1}^{\infty} a_k b_k$ converges.

□

No.10

1. **(10 marks)** Prove that
$$\sum_{k=1}^{\infty}(-1)^k \arcsin\left(\frac{1}{k}\right)$$
is conditionally convergent.

Solution. Firstly, we see that
$$\lim_{k\to\infty} \arcsin\left(\frac{1}{k}\right) = 0.$$

Next, let $f(x) = \arcsin\left(\frac{1}{x}\right)$ where $x > 1$. The derivative of $f(x)$ is

$$f'(x) = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} < 0 \quad \text{for all } x > 1.$$

So, $\left\{\arcsin\left(\frac{1}{k}\right)\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k \arcsin\left(\frac{1}{k}\right) \quad \text{converges.}$$

Finally, we consider
$$\sum_{k=1}^{\infty}\left|(-1)^k \arcsin\left(\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty} \arcsin\left(\frac{1}{k}\right)$$

and

$$\lim_{k\to\infty} \frac{\arcsin\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1 > 0$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty} \arcsin\left(\frac{1}{k}\right) \quad \text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \arcsin\left(\frac{1}{k}\right) \quad \text{is conditionally convergent.}$$

2. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k \sin\left(\frac{1}{k}\right)$$

is conditionally convergent.

Solution. Firstly, we see that

$$\lim_{k\to\infty} \sin\left(\frac{1}{k}\right) = 0.$$

Next, let $f(x) = \sin\left(\frac{1}{x}\right)$ where $x \geq 1$. By that fact that

$$0 < \frac{1}{k} \leq 1 < \frac{\pi}{2} \quad \text{for all } k \in \mathbb{N}, \text{ we obtain } \cos\left(\tfrac{1}{x}\right) > 0.$$

The derivative of $f(x)$ is

$$f'(x) = \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) < 0 \qquad \text{for all } x \geq 1.$$

So, $\left\{\sin\left(\frac{1}{k}\right)\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k \sin\left(\frac{1}{k}\right) \qquad \text{converges.}$$

Finally, we consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \sin\left(\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$$

and

$$\lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\cos\left(\frac{1}{k}\right)\cdot\left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k\to\infty} \cos\left(\frac{1}{k}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right) \qquad \text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \sin\left(\frac{1}{k}\right) \qquad \text{is conditionally convergent.}$$

3. **(10 marks)** Prove that

$$\sum_{k=1}^{\infty}(-1)^k \tan\left(\frac{1}{k}\right)$$

is conditionally convergent.

Solution. Firstly, we see that

$$\lim_{k\to\infty} \tan\left(\frac{1}{k}\right) = 0.$$

Next, let $f(x) = \tan\left(\frac{1}{x}\right)$ where $x \geq 1$. The derivative of $f(x)$ is

$$f'(x) = \sec^2\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) < 0 \quad \text{for all } x \geq 1.$$

So, $\left\{\tan\left(\frac{1}{k}\right)\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k \tan\left(\frac{1}{k}\right) \quad \text{converges.}$$

Finally, we consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \tan\left(\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty}\sin\left(\frac{1}{k}\right)$$

and

$$\lim_{k\to\infty}\frac{\tan\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty}\frac{\sec^2\left(\frac{1}{k}\right)\cdot\left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k\to\infty}\sec^2\left(\frac{1}{k}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty}\tan\left(\frac{1}{k}\right) \quad \text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \tan\left(\frac{1}{k}\right) \quad \text{is conditionally convergent.}$$