Final Examination

Suject	Mathematical Analysis MAP2406 **Semester** 2/2018
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University
Full Score	100 marks
Time	Wednesday 1 May 2019

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1. Use the **Mean Value Theorem** to prove that

$$\arctan x \le x \quad \text{for all } x \ge 0.$$

2. Determine whether f is **diferentiable** at $x = 0$ if

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3. Define $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ by

$$f(x) = \frac{1}{x + \cos x}$$

If f is 1-1 and continuous on $[-\frac{1}{2}, \frac{1}{2}]$. Use the **Inverse Function Theorem** to find $(f^{-1})'(1)$.

4. Use **Change of Variable** to show that $\displaystyle\int_{-\pi}^{\pi} \sin(x^3)\, dx = 0$.

5. If $f(x) = \displaystyle\int_{x^2}^{1} e^{\frac{1}{t}}\, dt$, show that

$$64 \int_1^2 \frac{f(x)}{x^5} dx - \int_1^4 e^{\frac{1}{x}} dx = 16(e^{\frac{1}{4}} - e).$$

6. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}$$

converges or **diverges**. Find the values if it converges .

7. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^{n+1} x^{n+1}}{3^{n+1}} = \left(\frac{x}{3+x}\right)^2$ where $|x| < 3$

8. Find the **interval of convergence** of $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

9. Suppose that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, **uniformly** on some set $E \subseteq \mathbb{R}$. Prove that

if f and g are **bounded** on E, then $f_n g_n \to fg$ **uniformly** on E.

10. Prove that the following limit exist and evaluate $\displaystyle\lim_{n\to\infty} \int_1^2 \frac{x^2 + n}{nx^3 + x}\, dx$.

11. Let $A \subseteq \mathbb{R}$ and A' be the **set of all limit points** of A. Prove that

A is closed **if and only if** $A = A \cup A'$.

12. Let $\{x_n\}$ be a sequence of real number. Prove that $\{x_n\}$ converges to $x \in \mathbb{R}$ **if and only if**

for every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Solution Final Examination

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1. Use the **Mean Value Theorem** to prove that

$$\arctan x \le x \quad \text{for all } x \ge 0.$$

Proof. Let $f(x) = \arctan x - x$ on $[0, x]$ where $x \ge 0$. Then f is continuous and differentiable on $[0, x]$. We obtain

$$f'(x) = \frac{1}{1+x^2} - 1 = -\frac{x^2}{1+x^2} \le 0 \quad \text{for all } x \ge 0.$$

By the Mean Value Theorem, there is a $c \in [0, x]$ such that

$$f(x) - f(0) = f'(c)(x - 0)$$
$$\arctan x - x = \left(-\frac{c^2}{1+c^2}\right) x \le 0$$

Since $x \ge 0$ and $-\frac{c^2}{1+c^2} \le 0$,

$$\arctan x \le x \quad \text{for all } x \ge 0$$

□

2. Determine whether f is **diferentiable** at $x = 0$ if

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$$

Since $0 \le \left|\sin\left(\frac{1}{x}\right)\right| \le 1$,

$$0 \le \left|x \sin\left(\frac{1}{x}\right)\right| \le |x|.$$

It follows that $\lim_{x\to 0} |x| = 0$. By Squeeze Theorem,

$$\lim_{x\to 0} \left|x \sin\left(\frac{1}{x}\right)\right| = 0.$$

So, $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. Therefore f is differentiable at $x = 0$ and

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

3. Define $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ by

$$f(x) = \frac{1}{x + \cos x}$$

If f is 1-1 and continuous on $[-\frac{1}{2}, \frac{1}{2}]$. Use the **Inverse Function Theorem** to find $(f^{-1})'(1)$.

Solution. Since

$$f(0) = \frac{1}{0 + \cos 0} = 1,$$

$f^{-1}(1) = 0$. Then

$$\begin{aligned} f'(x) &= -\frac{1}{(x + \cos x)^2} \cdot (1 - \sin x) \\ f'(0) &= -\frac{1}{(0 + \cos 0)^2} \cdot (1 - \sin 0) = -1 \end{aligned}$$

By the Inverse Function Theorem,

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-1} = -1 \quad \#$$

4. Use **Change of Variable** to show that $\int_{-\pi}^{\pi} \sin(x^3)\, dx = 0.$

Solution. Consider

$$\int_{-\pi}^{\pi} \sin(x^3)\, dx = \int_{-\pi}^{0} \sin(x^3)\, dx + \int_{0}^{\pi} \sin(x^3)\, dx.$$

Use Change of Variable $u(x) = -x$ to the term

$$\int_{-\pi}^{0} \sin(x^3)\, dx.$$

Hence,

$$\begin{aligned} \int_{-\pi}^{0} \sin(x^3)\, dx &= \int_{\pi}^{0} \sin((-u)^3)\, (-du) \\ &= \int_{0}^{\pi} \sin(-u^3)\, du \\ &= -\int_{0}^{\pi} \sin u^3\, du \\ &= -\int_{0}^{\pi} \sin x^3\, dx \end{aligned}$$

Therefore,

$$\int_{-\pi}^{\pi} \sin(x^3)\, dx = -\int_{0}^{\pi} \sin(x^3)\, dx + \int_{0}^{\pi} \sin(x^3)\, dx = 0.$$

5. If $f(x) = \int_{x^2}^{1} e^{\frac{1}{t}}\, dt$, show that

$$64 \int_{1}^{2} \frac{f(x)}{x^5} dx - \int_{1}^{4} e^{\frac{1}{x}}\, dx = 16(e^{\frac{1}{4}} - e).$$

Solution. Apply Chain rule and the Fundamental of Calculus,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \int_{x^2}^{1} e^{\frac{1}{t}}\, dt \\ &= \frac{d}{dx} \left(-\int_{1}^{x^2} e^{\frac{1}{t}}\, dt \right) \\ &= -e^{\frac{1}{x^2}} 2x = -2x e^{\frac{1}{x^2}} \end{aligned}$$

Use Integration by part we have

$$
\begin{aligned}
\int_1^2 \frac{f(x)}{x^5}\,dx &= \int_1^2 \left(-\frac{1}{4x^4}\right)' f(x)\,dx \\
&= \left[-\frac{1}{4x^4}x^3 f(x)\right]_1^2 - \int_1^2 \left(-\frac{1}{4x^4}\right) f'(x)\,dx \\
&= -\frac{1}{64}f(2) + \frac{1}{4}f(1) + \frac{1}{4}\int_1^2 \frac{1}{x^4}\left(-2xe^{\frac{1}{x^2}}\right)\,dx \\
&= -\frac{1}{64}\int_4^1 e^{\frac{1}{x}}\,dx + \frac{1}{4}\int_1^1 e^{\frac{1}{x}}\,dx - \frac{1}{2}\int_1^2 \frac{1}{x^3}\cdot e^{\frac{1}{x^2}}\,dx \\
&= \frac{1}{64}\int_1^4 e^{\frac{1}{x}}\,dx + 0 - \frac{1}{2}\int_1^2 \frac{1}{x^3}\cdot e^{\frac{1}{x^2}}\,dx \qquad ; u = \frac{1}{x^2} \\
&= \frac{1}{64}\int_1^4 e^{\frac{1}{x}}\,dx + \left[\frac{1}{4}e^{\frac{1}{x^2}}\right]_1^2 \\
&= \frac{1}{64}\int_1^4 e^{\frac{1}{x}}\,dx + \frac{1}{4}(e^{\frac{1}{4}} - e)
\end{aligned}
$$

Thus,

$$
64\int_1^2 \frac{f(x)}{x^5}dx - \int_1^4 e^{\frac{1}{x}}\,dx = 16(e^{\frac{1}{4}} - e).
$$

6. Determine whether the series

$$
\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}
$$

converges or **diverges**. Find the values if it converges .

Solution. Consider

$$
\begin{aligned}
\frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}} &= \frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1}+\sqrt{k})} \\
&= \frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1}+\sqrt{k})}\cdot\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}} \\
&= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}((\sqrt{k+1})^2 - (\sqrt{k})^2)} \\
&= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}} = \frac{\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}} - \frac{\sqrt{k}}{\sqrt{k}\sqrt{k+1}} \\
&= \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}
\end{aligned}
$$

We have

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}} \\
&= \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\
&= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\
&= 1 - \frac{1}{\sqrt{n+1}}
\end{aligned}
$$

Hence, the series converges and

$$
\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1.
$$

7. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}x^{n+1}}{3^{n+1}} = \left(\frac{x}{3+x}\right)^2$ where $|x| < 3$

Solution. Use geometric series (uniformly convergent)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{where } |x| < 1,$$

$$\frac{1}{1+\frac{x}{3}} = \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n \qquad \left|-\frac{x}{3}\right| < 1$$

$$\frac{3}{3+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n \qquad |x| < 3$$

$$\frac{d}{dx}\frac{3}{3+x} = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n \qquad |x| < 3$$

$$-\frac{3}{(3+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} n x^{n-1} \qquad |x| < 3$$

$$\frac{3x^2}{(3+x)^2} = -x^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} n x^{n-1} \qquad |x| < 3$$

$$-\frac{3}{(3+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} n x^{n-1} \qquad |x| < 3$$

$$\frac{3x^2}{(3+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} n x^{n+1} \qquad |x| < 3$$

$$-\frac{3}{(3+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} n x^{n-1} \qquad |x| < 3$$

$$\frac{x^2}{(3+x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^{n+1}} n x^{n+1} \qquad |x| < 3$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}x^{n+1}}{3^{n+1}} = \left(\frac{x}{3+x}\right)^2 \quad \text{where } |x| < 3$$

8. Find the **interval of convergence** of $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

Solution. Use the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^{2n}} \right| = |x^2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) \left(\frac{\ln n}{\ln(n+1)}\right)^2.$$

Apply L'Hospital's Rule to

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} \\ &= \lim_{x\to\infty} 1 + \frac{1}{x} = 1.\end{aligned}$$

So, $\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = 1$ and

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Then

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^{2n}} \right| = |x^2| \cdot 1 \cdot 1^2 = x^2.$$

So, the series converges when $|x^2| < 1$. That is $(-1, 1)$ to be an interval of convergence.
In this case $x = \pm 1$. Then

$$\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Let $f(x) = \frac{1}{x(\ln x)^2}$ where $x \geq 2$. We obtain

$$\begin{aligned} f'(x) &= -\frac{1}{(x(\ln x)^2)^2} \cdot \left(x 2 \ln x \cdot \frac{1}{x} + 1 \cdot (\ln x)^2 \right) \\ &= -\frac{1}{(x(\ln x)^2)^2} \cdot \left(2 \ln x \cdot + (\ln x)^2 \right) < 0 \qquad \text{for all } x \geq 2 \end{aligned}$$

So, f is decreasing. That is $\left\{ \frac{1}{n(\ln n)^2} \right\}$ to be decreasing. Then

$$\begin{aligned} \int_2^{\infty} f(x) dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2} dx \\ &= \lim_{t\to\infty} \left[-\frac{1}{\ln x} \right]_2^t \qquad ; u = \frac{1}{x} \\ &= \lim_{t\to\infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] \\ &= \frac{1}{\ln 2} \end{aligned}$$

Hence, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. Therefore, $[-1, 1]$ is the interval of convergent of the series.

9. Suppose that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, **uniformly** on some set $E \subseteq \mathbb{R}$. Prove that

if f and g are **bounded** on E, then $f_n g_n \to fg$ **uniformly** on E.

Proof. Suppose that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, uniformly on E and f and g are **bounded** on E. There are $M > 0$ and $L > 0$ such that

$$|f(x)| \leq L \quad \text{and} \quad |g(x)| \leq M \quad \text{for all } x \in E.$$

Let $\varepsilon > 0$. Then there are $N_1, N_2, N_3 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\quad \longrightarrow \quad |f_n(x) - f(x)| < 1 \\ n \geq N_2 &\quad \longrightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2M} \\ n \geq N_3 &\quad \longrightarrow \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2(L+1)} \end{aligned}$$

From $|f_n(x)| - |f(x)| \leq |f_n(x) - f(x)| < 1$, it follows that

$$|f_n(x)| \leq 1 + |f(x)| \leq 1 + L.$$

Choose $N = \max\{N_1, N_2, N_3\}$. For each $n \geq N_3$ and $x \in E$, we obtain

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &\leq (L+1) \cdot \frac{\varepsilon}{2(L+1)} + M \cdot \frac{\varepsilon}{2M} = \varepsilon \end{aligned}$$

Thus, $f_n g_n \to fg$ is uniformly on E.

□

10. Prove that the following limit exist and evaluate $\displaystyle\lim_{n\to\infty} \int_1^2 \frac{x^2 + n}{nx^3 + x}\, dx$.

Solution. Let $f_n(x) = \dfrac{x^2 + n}{nx^3 + x}$ where $x \in [1, 2]$.

Let $\varepsilon > 0$. By Archimedean Principle, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{15}$.

Since $1 \leq x \leq 2$, $1 \leq x^4 \leq 16$ and $1 \leq x^5 \leq 32$. That is

$$0 \leq x^4 - 1 \leq 15 \quad \text{and} \quad \frac{1}{32} \leq \frac{1}{x^5} \leq 1$$

For $n \geq N$ and $x \in [1, 2]$, we have $\dfrac{1}{n} \leq \dfrac{1}{N}$ and

$$\begin{aligned} \left| f_n(x) - \frac{1}{x^3} \right| &= \left| \frac{x^2 + n}{nx^3 + x} - \frac{1}{x^3} \right| = \left| \frac{x^5 - x}{x^3(nx^3 + x)} \right| = \left| \frac{x^4 - 1}{x^3(nx^2 + 1)} \right| \\ &\leq \frac{15}{x^3(nx^2 + 1)} \leq \frac{15}{x^3(nx^2)} = \frac{15}{x^5 n} = \frac{15}{n} \cdot \frac{1}{x^5} \\ &\leq \frac{15}{n} \leq \frac{15}{N} < \varepsilon \end{aligned}$$

So, $f_n(x) \to \dfrac{1}{x^3}$ converges uniformly on $[1, 2]$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_1^2 \frac{x^2 + n}{nx^3 + x}\, dx &= \int_1^2 \lim_{n\to\infty} \frac{x^2 + n}{nx^3 + x}\, dx \\ &= \int_1^2 \frac{1}{x^3}\, dx \\ &= \left[-\frac{1}{2x^2} \right]_1^2 \\ &= -\frac{1}{8} + \frac{1}{2} = \frac{3}{8} \quad \# \end{aligned}$$

11. Let $A \subseteq \mathbb{R}$ and A' be the set of all limit points of A. Prove that

$$A \text{ is closed} \quad \textbf{if and only if} \quad A = A \cup A'.$$

Proof. Suppose that A is closed. Obviously, $A \subseteq A \cup A'$. It remain to show that $A \cup A' \subseteq A$.
Let $x \in A \cup A'$.
Case $x \in A$. It's done.
Case $x \in A'$. Suppose that $x \notin A$. Then $x \in A^c$. Since A^c is open, there is a $\delta > 0$ such that

$$(x - \delta, x + \delta) \subseteq A^c.$$

So, $(x - \delta, x + \delta) \cap A = \varnothing$. It follows that

$$(x - \delta, x + \delta) \cap A - \{x\} = \varnothing \quad \text{since } x \notin A.$$

Hence, x is not a limit point of A. It contradicts.
Conversely, assume that $A = A \cup A'$. Let $x \in A^c$. Then $x \notin A$. We obtain

$$x \notin A \cup A', \text{ i.e., } x \notin A \text{ and } x \notin A'$$

So, there is $\delta > 0$ such that

$$(x - \delta, x + \delta) \cap A - \{x\} = \varnothing.$$

Then, $(x - \delta, x + \delta) \cap A = \varnothing$. Hence,

$$(x - \delta, x + \delta) \subseteq A^c.$$

Therefore, A^c is open. □

12. Let $\{x_n\}$ be a sequence of real number. Prove that $\{x_n\}$ converges to $x \in \mathbb{R}$ **if and only if**

for every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Proof. Assume that $x_n \to x$. Let U be a neighborhood of x. Then there is a $\delta > 0$ such that

$$(x - \delta, x + \delta) \subseteq U.$$

There is an $N \in \mathbb{N}$ and $n \geq N$, it implies that

$$\begin{aligned} |x_n - x| &< \delta \\ -\delta < x_n - x &< \delta \\ x - \delta < x_n &< x + \delta \end{aligned}$$

Thus, $x_n \in (x - \delta, x + \delta) \subseteq U$ for all $n > N$.
Conversely, assume that every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.
Let $\varepsilon > 0$. Then $U := (x - \varepsilon, x + \varepsilon)$ is a neighborhood of x (it is clear). By assumption,

there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$

So, $x_n \in (x - \varepsilon, x + \varepsilon)$, i.e.,

$$\begin{aligned} x - \varepsilon < x_n &< x + \varepsilon \\ -\varepsilon < x_n - x &< \varepsilon \\ |x_n - x| &< \varepsilon \end{aligned}$$

Hence, $x_n \to x$.

□