Math SSRU

Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Midterm Examination
Semester 2/2022

Course ID	Course Name	Test Time	Full Scores
MAC3309	Mathematical Analysis	5pm - 8pm Mon 6 Feb 2023	100 marks 25%

Name.. **ID**.................................... **Section**...............

Direction

1. 10 questions of all 10 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
4. Cannot answer sheets out of test room.
5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

1. **(10 marks)** Let a and b be real numbers. Prove that

$$(a+b+1)^2 \leq 3(a^2+b^2+1).$$

2. **(10 marks)** Let x be a real numbers. Prove that

$$|1-x| = 1+|x| \quad \text{if and only if} \quad |x|+x=0.$$

3. **(10 marks)** Define the set

$$A = \left\{ 1 + \frac{2n}{n+1} : n \in \mathbb{N} \right\}.$$

Find **sup *A*** and **inf *A*** with proving them.

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{n(n+1)}{n^2+1} = 1.$$

5. **(10 marks)** Assume that $x_n \to 1$ as $n \to \infty$. Show that

$$\frac{x_n}{n^2} \to 0 \quad \text{as} \quad n \to \infty.$$

6. **(10 marks)** Use definition to prove that

$$\{\sqrt{n+1} - \sqrt{n}\} \quad \text{is a Caucy sequence.}$$

7. **(10 marks)** Define a set

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Determine whether 0 is a **limit point** of E. Verify your answer.

8. **(10 marks)** Use definition to prove that

$$\lim_{x \to 2} \frac{2x}{x-3} = -4.$$

9. **(10 marks)** Use definition to prove that

$$\lim_{x\to 1^-} \frac{1}{x^2-1} = -\infty.$$

10. **(10 marks)** Let f and g be real functions from a set E to $\mathbb{R}$.
Assume that there are $\delta_0 > 0$ and $K > 0$ such that

$$|g(x)| \leq K \quad \text{for all } x \in (a - \delta_0, a + \delta_0) \subseteq E.$$

Let a be a limit point of E and $f(x) > 0$ on E. Prove that if $f(x) \to \infty$ as $x \to a$, then

$$\frac{g(x)}{f(x)} \to 0 \text{ as } x \to a.$$

Math SSRU

Solution Midterm Exam. 2/2022
MAC3309 Mathematical Analysis

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1. **(10 marks)** Let a and b be real numbers. Prove that

$$(a+b+1)^2 \leq 3(a^2+b^2+1).$$

Proof. Let a and b be real numbers. By the fact that

$$(a-b)^2 \geq 0,\ (a-1)^2 \geq 0 \text{ and } (b-1)^2 \geq 0.$$

We obtain

$$\begin{aligned} 0 &\leq (a-b)^2 + (a-1)^2 + (b-1)^2 \\ 0 &\leq (a^2-2ab+b^2) + (a^2-2a+1) + (b^2-2b+1) \\ 2ab+2a+2b &\leq 2a^2+2b^2+2 \\ (a^2+b^2+1)+2ab+2a+2b &\leq 2a^2+2b^2+2+(a^2+b^2+1) \\ (a+b+1)^2 &\leq 3(a^2+b^2+1) \end{aligned}$$

□

2. **(10 marks)** Let x be a real numbers. Prove that

$$|1-x| = 1+|x| \quad \text{if and only if} \quad |x|+x=0.$$

Proof. Let x be a real numbers.
Assume that $|1-x| = 1+|x|$. Then

$$\begin{aligned} |1-x|^2 &= (1+|x|)^2 \\ (1-x)^2 &= 1+2|x|+|x|^2 \\ 1-2x+x^2 &= 1+2|x|+x^2 \\ -x &= |x|. \end{aligned}$$

So, $|x|+x=0$.
Conversely, we assume that $|x|+x=0$. We obatin

$$\begin{aligned} -x &= |x| \\ 1-x &= 1+|x| \end{aligned}$$

Since $|x| \geq 0$, $1+|x| \geq 1 > 0$. So, $1-x > 0$. Thus,

$$1-x = |1-x| = 1+|x|.$$

□

3. **(10 marks)** Define the set

$$A = \left\{ 1 + \frac{2n}{n+1} : n \in \mathbb{N} \right\}.$$

Find **sup $\boldsymbol{A}$** and **inf $\boldsymbol{A}$** with proving them.

Claim that $\inf A = 2$ and $\sup A = 3$

Proof. $\inf A = 2$

Let $n \in \mathbb{N}$. Then $1 \leq n$. So, $1 + n \leq n + n = 2n$. It's clear that $1 \leq \frac{2n}{n+1}$. We obtain

$$2 = 1 + 1 \leq 1 + \frac{2n}{n+1}.$$

Thus, 2 is a lower bound of A.
Let ℓ be a lower bound of A. For $n = 1$, we get

$$2 = 1 + \frac{2(1)}{1+1} \in A.$$

So, $\ell \leq 2$. Hence, $\inf A = 2$.

$\sup A = 3$

Let $n \in \mathbb{N}$. It is easy to see that $2n < 2 + 2n = 2(1+n)$. Then $\frac{2n}{n+1} \leq 2$. We have

$$1 + \frac{2n}{n+1} \leq 1 + 2 = 3.$$

Thus, 3 is an upper bound of A.

Assume that that there is an upper bound u_0 of A such that

$$u_0 < 3.$$

By definition,

$$1 + \frac{2n}{n+1} \leq u_0 \quad \text{for all } n \in \mathbb{N} \qquad (*)$$

Since $u_0 < 3$, $\frac{3-u_0}{2} > 0$. By Archimendean property, there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{3-u_0}{2}.$$

By the fact that $n_0 + 1 > n_0$,

$$\frac{1}{n_0+1} < \frac{1}{n_0} < \frac{3-u_0}{2}$$
$$u_0 < 3 - \frac{2}{n_0+1} = \frac{3n_0+1}{n_0+1} = 1 + \frac{2n_0}{n_0+1}.$$

This is contradiction to $(*)$. Therefore, $\sup A = 3$. □

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{n(n+1)}{n^2+1} = 1.$$

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{1}{n} \leq \frac{1}{N}$. Since $n^2+1 > n^2$, $\frac{1}{n^2+1} < \frac{1}{n^2}$. From $n \geq 1$, we have $|n-1| = n-1 > n$. It follows that

$$\begin{aligned}\left|\frac{n(n+1)}{n^2+1} - 1\right| &= \left|\frac{n(n+1)-(n^2+1)}{n^2+1}\right| \\ &= |n-1| \cdot \frac{1}{n^2+1} < n \cdot \frac{1}{n^2} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n(n+1)}{n^2+1} = 1$. □

5. **(10 marks)** Assume that $x_n \to 1$ as $n \to \infty$. Show that

$$\frac{x_n}{n^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$.
Given $\varepsilon = 1$. There is an $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - 1| < 1$. Then

$$\begin{aligned}|x_n| - 1 \leq |x_n - 1| &\leq 1 \\ |x_n| &\leq 2.\end{aligned}$$

Let $\varepsilon > 0$. By Archimedean property, there is an $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\varepsilon}{2}$.
Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain $n^2 > N^2$. So, $\frac{1}{n^2} \leq \frac{1}{N^2}$.
Since $N \geq N_1 > 0$, $N^2 \geq N^2$. We have $\frac{1}{N^2} < \frac{1}{N_1^2}$. Use $N_1^2 \geq N_1$ to obtain $\frac{1}{N_1^2} \leq \frac{1}{N_1}$.
It follows that

$$\left|\frac{x_n}{n^2} - 0\right| = |x_n| \cdot \frac{1}{n^2} \leq 2 \cdot \frac{1}{N^2} \leq 2 \cdot \frac{1}{N_1^2} \leq 2 \cdot \frac{1}{N_1} = \varepsilon.$$

Thus, $\frac{x_n}{n^2} \to 0$ as $n \to \infty$. □

6. **(10 marks)** Use definition to prove that

$$\{\sqrt{n+1}-\sqrt{n}\} \quad \text{is a Caucy sequence.}$$

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon^2}{4}$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\sqrt{n} > \sqrt{N}$ and $\sqrt{m} > \sqrt{N}$.
We obatin $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}$ and $\frac{1}{\sqrt{m}} \leq \frac{1}{\sqrt{N}}$. It follows that

$$\begin{aligned}
\left|(\sqrt{n+1}-\sqrt{n})-(\sqrt{m+1}-\sqrt{m})\right| &= \left|(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}-(\sqrt{m+1}-\sqrt{m})\cdot\frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}+\sqrt{m}}\right| \\
&= \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}-\frac{1}{\sqrt{m+1}+\sqrt{m}}\right| \\
&\leq \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right|+\left|\frac{1}{\sqrt{m+1}+\sqrt{m}}\right| \\
&\leq \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}} \\
&\leq \frac{1}{\sqrt{N}}+\frac{1}{\sqrt{N}} = \frac{2}{\sqrt{N}} < \varepsilon.
\end{aligned}$$

Thus, $\{\sqrt{n+1}-\sqrt{n}\}$ is Cauchy.

□

7. **(10 marks)** Define a set

$$E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Determine whether 0 is a **limit point** of E. Verify your answer.

Answer : 0 is a limit point of E.

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
So, $0 < \frac{1}{N} < \varepsilon$. We see that

$$\frac{1}{N} \in (0, \varepsilon) \quad \text{and} \quad \frac{1}{N} \in E.$$

It implies that

$$[(-\varepsilon, 0) \cup (0, \varepsilon)] \cap E \neq \varnothing.$$

Therefore, 0 is a limit point of E. □

8. **(10 marks)** Use definition to prove that

$$\lim_{x \to 2} \frac{2x}{x-3} = -4.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{0.5, \frac{\varepsilon}{12}\right\}$. Suppose that $0 < |x-2| < \delta$. Then $0 < |x-2| < 0.5$. We have

$$\begin{aligned} -0.5 &< x-2 < 0.5 \\ -1.5 &< x-3 < -0.5 \\ 0.5 &< |x-3| < 1.5. \end{aligned}$$

We get $\frac{1}{|x-3|} < 2$. Then,

$$\left|\frac{2x}{x-3} + 4\right| = \left|\frac{6x-12}{x-3}\right| = \left|\frac{6(x-2)}{x-3}\right| = 6 \cdot |x-2| \cdot \frac{1}{|x-3|} < 6\delta \cdot 2 < \varepsilon.$$

Therefore, $\lim_{x \to 2} \frac{2x}{x-3} = -4$. □

9. **(10 marks)** Use definition to prove that

$$\lim_{x \to 1^-} \frac{1}{x^2-1} = -\infty.$$

Proof. Let $M < 0$. Choose $\delta = \min\left\{1, -\frac{1}{M}\right\}$. Then $\delta > 0$, $\delta \leq 1$ and $\delta \leq -\frac{1}{M}$.
Let $x \in \mathbb{R}$ such that $-\delta < x-1 < 0$. Then $-1 < x-1 < 0$. So, $1 < x+1 < 2$.
We obatin

$$\frac{1}{x-1} < -\frac{1}{\delta} \quad \text{and} \quad \frac{1}{2} < \frac{1}{x+1} < 1$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} < -\frac{1}{\delta} \cdot 1 < M$$

Thus, $\lim_{x \to 1^-} \frac{1}{x^2-1} = -\infty$. □

10. **(10 marks)** Let f and g be real functions from a set E to $\mathbb{R}$.
Assume that there are $\delta_0 > 0$ and $K > 0$ such that

$$|g(x)| \leq K \qquad \text{for all } x \in (a - \delta_0, a + \delta_0) \subseteq E.$$

Let a be a limit point of E and $f(x) > 0$ on E. Prove that if $f(x) \to \infty$ as $x \to a$, then

$$\frac{g(x)}{f(x)} \to 0 \text{ as } x \to a.$$

Proof. Assume that there are $\delta_0 > 0$ and $K > 0$ such that

$$|g(x)| \leq K \qquad \text{for all } x \in (a - \delta_0, a + \delta_0) \subseteq E.$$

Let a be a limit point of E and $f(x) > 0$ on E. Suppose that $f(x) \to \infty$ as $x \to a$.
Let $\varepsilon > 0$. Then $M := \dfrac{K}{\varepsilon} > 0$. There is a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \text{implies} \quad f(x) > M = \frac{K}{\varepsilon}.$$

Choose $\delta = \min\{\delta_0, \delta_1\}$. Let $x \in E$ such that $0 < |x - a| < \delta$.
We obtain $x \in (a - \delta_0, a + \delta_0)$ and $0 < |x - a| < \delta_1$. So,

$$|g(x)| \leq K \quad \text{and} \quad \frac{1}{f(x)} < \frac{\varepsilon}{K}.$$

It follows that

$$\left|\frac{g(x)}{f(x)} - 0\right| = |g(x)| \cdot \frac{1}{f(x)} < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Thus, $\dfrac{g(x)}{f(x)} \to 0$ as $x \to a$. $\square$