Midterm Examination

Suject	Mathematical Analysis MAP2406 **Semester** 2/2018
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Full Score	100 marks
Time	Friday 8 March 2019

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1. Prove that
$$\frac{a}{a^2+1} \leq \frac{1}{2} \quad \text{for all } a \in \mathbb{R}.$$

2. Let $x, y, z, w \in \mathbb{R}$. Show that
$$(xy+zw)^2 \leq (x^2+z^2)(y^2+w^2).$$

3. Use mathematical induction to prove that
$$\sum_{k=1}^{n} k \cdot 2^k = 2^{n+1}(n-1)+2 \quad \text{for all } n \in \mathbb{N}.$$

4. Find $\inf A$ and prove it if
$$A = \left\{ \frac{1}{n^2+1} : n \in \mathbb{Z} \right\}.$$

5. Use definition to prove that $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1.$

6. Suppose that x_n is sequence of real numbers that converges to 1 as $n \to \infty$. Use definition to prove that
$$x_n^2 + 1 \to 2 \quad \text{as} \quad n \to \infty.$$

7. Prove that every Cauchy sequence in $\mathbb{R}$ is bounded.

8. Prove that a sequence $\left\{ \frac{1}{n} \right\}$ is Cauchy.

9. Use definition to prove that
$$\lim_{x\to 1^+} \sqrt{x^2-1} = 0.$$

10. Let f and g be functions with continuous at a. Prove that $f+g$ is continuous at a.

11. Use definition to prove that
$$\lim_{x\to 2} \frac{x^2+1}{x-1} = 5.$$

12. Prove that $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$.

Solution Midterm Examination

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1. Prove that

$$\frac{a}{a^2+1} \le \frac{1}{2} \quad \text{for all } a \in \mathbb{R}.$$

Proof. Let $a \in \mathbb{R}$. Consider $(a-1)^2 \ge 0$. So,

$$\begin{aligned} (a-1)^2 &\ge 0 \\ a^2 - 2a + 1 &\ge 0 \\ a^2 + 1 &\ge 2a \\ \frac{1}{2} &\ge \frac{a}{a^2+1} && \because a^2 + 1 > 0 \end{aligned}$$

Thus, $\frac{a}{a^2+1} \le \frac{1}{2}$. □

2. Let $x, y, z, w \in \mathbb{R}$. Show that

$$(xy + zw)^2 \le (x^2 + z^2)(y^2 + w^2).$$

Proof. $x, y, z, w \in \mathbb{R}$. Then

$$\begin{aligned} (xy+zw)^2 &= x^2y^2 + 2xyzw + z^2w^2 \\ &= x^2y^2 + 2xyzw + z^2w^2 + x^2w^2 - x^2w^2 + z^2y^2 - z^2y^2 \\ &= (x^2y^2 + x^2w^2) + (z^2w^2 + z^2y^2) - (x^2w^2 - 2xyzw + z^2y^2) \\ &= x^2(y^2 + w^2) + z^2(w^2 + y^2) - (xw - zy)^2 \\ &\le (x^2 + z^2)(y^2 + w^2) && \because (xw - zy)^2 \ge 0 \end{aligned}$$

□

3. Use mathematical induction to prove that

$$\sum_{k=1}^{n} k \cdot 2^k = 2^{n+1}(n-1) + 2 \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let $P(n)$ represent the statement

$$\sum_{k=1}^{n} k \cdot 2^k = 2^{n+1}(n-1) + 2 \quad \text{for all } n \in \mathbb{N}.$$

Since $1 \cdot 2^1 = 2 = 0 + 2 = 4 \cdot 0 + 2 = 2^{1+1}(1-1) + 2$, $P(1)$ is true.

Let $n \in \mathbb{N}$. Assume that $P(n)$ is true. Then $\sum_{k=1}^{n} k \cdot 2^k = 2^{n+1}(n-1) + 2$. We obtain

$$\begin{aligned}
\sum_{k=1}^{n+1} k \cdot 2^k &= \sum_{k=1}^{n} k \cdot 2^k + (n+1)2^{n+1} \\
&= 2^{n+1}(n-1) + 2 + (n+1)2^{n+1} \\
&= 2^{n+1}(n-1+n+1) + 2 \\
&= 2^{n+1}(2n) + 2 \\
&= 2^{n+2}(n) + 2
\end{aligned}$$

So, $P(n+1)$ is true. We conclue by inductuion that $P(n)$ holds for all $n \in \mathbb{N}$.

□

4. Find $\inf A$ and prove it if

$$A = \left\{ \frac{1}{n^2+1} : n \in \mathbb{Z} \right\}.$$

Consider

$$A = \left\{ 1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots \right\}$$

Claim that $\inf A = 0$.

Proof. It is easy to see that

$$0 \le \frac{1}{n^2+1} \quad \text{for all } n \in \mathbb{Z}.$$

So, 0 is a lower bound of A.

Suppose that there a lower bound ℓ of A such that $0 < \ell$. Then $\sqrt{\ell} > 0$. By Archimidean priciple, $\exists N \in \mathbb{N}$ such that

$$\frac{1}{N} < \sqrt{\ell}$$

We obtain

$$\frac{1}{N^2+1} < \frac{1}{N^2} < \ell$$

So, ℓ is not lower bound of A. It is contradiction. □

5. Use definition to prove that $\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1.$

Proof. Let $\varepsilon > 0$. Then $\frac{1}{\sqrt{\frac{2}{\varepsilon}+1}} > 0$. By Archimidean priciple, $\exists N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{1}{\sqrt{\frac{2}{\varepsilon}+1}}$$

Then $N > \sqrt{\frac{2}{\varepsilon}+1} > 1$ and

$$\begin{aligned} N^2 &> \frac{2}{\varepsilon}+1 \\ N^2-1 &> \frac{2}{\varepsilon} > 0 \\ \frac{1}{N^2-1} &< \frac{\varepsilon}{2} \end{aligned}$$

For each $n \geq N > 1$, i.e., $n^2-1 \geq N^2-1 > 0$. So, $\frac{1}{n^2-1} \leq \frac{1}{N^2-1}$. Then

$$\begin{aligned} \left|\frac{n^2+1}{n^2-1} - 1\right| &= \left|\frac{2}{n^2-1}\right| \\ &\leq \frac{2}{N^2-1} < \varepsilon. \end{aligned}$$

□

6. Suppose that x_n is sequence of real numbers that converges to 1 as $n \to \infty$. Use definition to prove that

$$x_n^2 + 1 \to 2 \quad \text{as} \quad n \to \infty.$$

Proof. Suppose that x_n converges to 1 as $n \to \infty$. Since x_n is convergent, x_n is bounded. Then $\exists M > 0$ such that

$$|x_n| < M \qquad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$, $n \geq N$

$$|x_n - 1| < \frac{\varepsilon}{M+1}$$

For each $n \geq N$, we obtain

$$\begin{aligned} |x_n^2 - 1| &= |(x_n-1)(x_n+1)| \\ &= |x_n-1||x_n+1| \\ &< \frac{\varepsilon}{M+1}(|x_n|+1) \\ &< \frac{\varepsilon}{M+1}(M+1) = \varepsilon \end{aligned}$$

□

7. Prove that every Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Suppose that $\{x_n\}$ is Cuachy. Then

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N},\ n, m \geq N \text{ implies } |x_n - x_m| < \varepsilon$$

Choose $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ $m, n \geq 2N$ implies $|x_n - x_m| < 1$.
Choose $m = N$. Then $|x_n - x_N| < 1$, i.e.,

$$|x_n| < 1 + |x_N| \quad \text{ for all } n \geq N.$$

□

In case $n = 1, 2, 3, ..., N-1$, we can choose the maximum value of $|x_1|, |x_2|, |x_3|, ..., |x_{N-1}|$.
Thus, set

$$M = \max\{|x_1|, |x_2|, |x_3|, ..., |x_{N-1}|, 1 + |x_N|\}.$$

So,

$$|x_n| \leq M \quad \text{ for all } n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is bounded.

8. Prove that a sequence $\left\{\frac{1}{n}\right\}$ is Cauchy.

Proof. Let $\varepsilon > 0$. By Archimedean principle, $\exists\, N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$.
For each $m, n \geq N$, we have $\frac{1}{n} < \frac{1}{N}$ and $\frac{1}{m} < \frac{1}{N}$. Then

$$\begin{aligned} |x_n - x_m| &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{1}{N} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, $\left\{\frac{1}{n}\right\}$ is Cauchy. □

9. Use definition to prove that

$$\lim_{x \to 1^+} \sqrt{x^2 - 1} = 0.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{\varepsilon^2}{4}, 2\right\}$. Suppose $0 < x - 1 < \delta$.
Then $0 < x - 1 < 2$, i.e., $2 < x + 1 < 4$. We obtain

$$\begin{aligned} |\sqrt{x^2 - 1} - 0| &= \sqrt{(x-1)(x+1)} \\ &= \sqrt{(x-1)} \cdot \sqrt{(x+1)} \\ &< \sqrt{\delta} \cdot \sqrt{4} \\ &< \sqrt{\frac{\varepsilon^2}{4}} \cdot 2 = \varepsilon \end{aligned}$$

□

10. Let f and g be functions with continuous at a. Prove that $f+g$ is continuous at a.

Proof. Suppose f and g be functions with continuous at a.
Let $\varepsilon > 0$. there are positive numbers δ_1 and δ_2 such that

$$|x-a| < \delta_1 \quad \text{imples } |f(x) - f(a)| < \frac{\varepsilon}{2}$$
$$|x-a| < \delta_2 \quad \text{imples } |g(x) - g(a)| < \frac{\varepsilon}{2}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. If $|x-a| < \delta$, it implies that

$$\begin{aligned} |(f+g)(x) - (f+g)(a)| &= |(f(x) - f(a)) + (g(x) - g(a))| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, $f+g$ is continuous at a.

□

11. Use definition to prove that

$$\lim_{x \to 2} \frac{x^2+1}{x-1} = 5.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{\varepsilon}{6}, 1\right\}$. Suppose $0 < |x-2| < \delta$. Then

$$\begin{aligned} |x-2| &< 1 \\ |x| - 2 &< 1 \\ |x| &< 3 \end{aligned}$$

and

$$\begin{aligned} 0 < |x-2| &< 1 && \\ -1 < x-2 &< 1 && \text{when } x \neq 2 \\ 1 < x-1 &< 3 && \text{when } x \neq 2 \\ \frac{1}{3} < \frac{1}{x-1} &< 1 && \text{when } x \neq 2 \end{aligned}$$

$0 < x- < 1$, i.e., $2 < x+1 < 4$. We obtain

$$\begin{aligned} \left|\frac{x^2+1}{x-1} - 5\right| &= \left|\frac{x^2-5x+6}{x-1}\right| \\ &= \left|\frac{(x-2)(x-3)}{x-1}\right| \\ &= |x-2| \cdot |x-3| \cdot \frac{1}{|x-1|} \\ &< \delta \cdot (|x|+3) \cdot 1 \\ &< \delta \cdot (3+3) \\ &< \frac{\varepsilon}{6} \cdot 6 = \varepsilon \end{aligned}$$

□

12. Prove that $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Let $x, a \in\in \mathbb{R}$ such that $|x - a| < \delta$. Then

$$\begin{aligned}
|\cos x - \cos a| &= \left|2 \sin\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right)\right| \\
&\leq 2 \cdot 1 \cdot \left|\sin\left(\frac{x-a}{2}\right)\right| \\
&\leq 2 \cdot 1 \cdot \left|\frac{x-a}{2}\right| \\
&\leq |x - a| < \varepsilon
\end{aligned}$$

Thus, f is uniformly continuous on $\mathbb{R}$. □