Suan Sunandha Rajabhat University
Faculty of Education
Division of Mathematics
Midterm Examination Semester 2/2021

Subject	Mathematical Analysis
ID	MAC3310
Place	Zoom
Time	1 p.m. (3 hours) Thursday 20 January 2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.
Marks	100 (25%)

No.1

1. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+2}{\sqrt{a^2+1}} \geq 2.$$

2. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+3}{\sqrt{a^2+2}} \geq 2.$$

3. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+4}{\sqrt{a^2+3}} \geq 2.$$

4. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+5}{\sqrt{a^2+4}} \geq 2.$$

No.2

1. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x+y| = |x-y|, \quad \text{then} \quad x|y| + y|x| = 0.$$

2. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x+y| = |x+2y|, \quad \text{then} \quad |xy| = x^2.$$

3. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x-y| = |x-2y|, \quad \text{then} \quad |xy| = x^2.$$

4. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x+y| = |x+2y|, \quad \text{then} \quad |xy| = y^2.$$

No.3

1. **(10 marks)** Let $A = \left\{ 1 - \dfrac{n}{n^2+1} : n \in \mathbb{N} \right\}$.
   What are **supremum** and **infimum** of A ? Verify (proof) your answers.

2. **(10 marks)** Let $A = \left\{ 1 - \dfrac{n}{n^2+2} : n \in \mathbb{N} \right\}$.
   What are **supremum** and **infimum** of A ? Verify (proof) your answers.

3. **(10 marks)** Let $A = \left\{ 2 - \dfrac{n}{n^2+1} : n \in \mathbb{N} \right\}$.
   What are **supremum** and **infimum** of A ? Verify (proof) your answers.

4. **(10 marks)** Let $A = \left\{ 2 - \dfrac{n}{n^2+2} : n \in \mathbb{N} \right\}$.
   What are **supremum** and **infimum** of A ? Verify (proof) your answers.

No.4

1. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+2)}{n^2+1} \quad \text{exists.}$$

2. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+3)}{n^2+2} \quad \text{exists.}$$

3. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+3)}{n^2+1} \quad \text{exists.}$$

4. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+5)}{n^2+2} \quad \text{exists.}$$

No.5

1. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+1}) = 0.$$

2. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+2}) = 0.$$

3. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+3}) = 0.$$

4. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+4}) = 0.$$

No.6

1. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

2. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

3. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^3} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

4. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^4} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

No.7

1. **(10 marks)** Let F be a closed set. Assume that $\{x_n\}$ is a sequence in F (it means that $x_n \in F$ for all $n \in \mathbb{N}$). Prove that

$$\text{if} \quad x_n \to a \text{ as } n \to \infty, \quad \text{then} \quad a \in F.$$

No.8

1. **(10 marks)** Use definition to prove that

$$\lim_{x \to 1} \left(x + \frac{1}{x} \right) = 2.$$

2. **(10 marks)** Use definition to prove that

$$\lim_{x \to -1} \left(x + \frac{1}{x} \right) = 0.$$

3. **(10 marks)** Use definition to prove that

$$\lim_{x \to 2} \left(x + \frac{2}{x} \right) = 3.$$

4. **(10 marks)** Use definition to prove that

$$\lim_{x \to -2} \left(x + \frac{2}{x} \right) = -3.$$

No.9

1. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x\to 1^+} (x + f(x)) = +\infty.$$

Prove that $f(x) \to +\infty$ as $x \to 1^+$.

2. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x\to 1^-} (x + f(x)) = +\infty.$$

Prove that $f(x) \to +\infty$ as $x \to 1^-$.

3. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x\to 1^+} (x + f(x)) = -\infty.$$

Prove that $f(x) \to -\infty$ as $x \to 1^+$.

4. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x\to 1^-} (x + f(x)) = -\infty.$$

Prove that $f(x) \to -\infty$ as $x \to 1^-$.

No.10

1. **(10 marks)** Use definition (sequence) to prove that

$$\lim_{n\to\infty}\left(1-\sqrt{n}\right) = -\infty.$$

2. **(10 marks)** Use definition (sequence) to prove that

$$\lim_{n\to\infty}\left(\sqrt{n}-1\right) = \infty.$$

3. **(10 marks)** Use definition (function) to prove that

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = 0.$$

4. **(10 marks)** Use definition (function) to prove that

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1}+x\right) = 0.$$

Solution Midterm : MAC3309 Mathematical Analysis

No.1

1. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+2}{\sqrt{a^2+1}} \geq 2.$$

Proof. Let $a \in \mathbb{R}$. By the fact that $a^2 \geq 0$, we obtain

$$\begin{aligned}
[(a^2+2)-2]^2 &\geq 0 \\
(a^2+2)^2 - 4(a^2+2) + 4 &\geq 0 \\
(a^2+2)^2 &\geq 4(a^2+2) - 4 \\
(a^2+2)^2 &\geq 4(a^2+2-1) \\
(a^2+2)^2 &\geq 4(a^2+1) \\
\sqrt{(a^2+2)^2} &\geq 2\sqrt{a^2+1} \\
|a^2+2| &\geq 2\sqrt{a^2+1} \\
\frac{a^2+2}{\sqrt{a^2+1}} &\geq 2
\end{aligned}$$

□

2. **(10 marks)** Let $a \in \mathbb{R}$. Prove that
$$\frac{a^2+3}{\sqrt{a^2+2}} \geq 2.$$

Proof. Let $a \in \mathbb{R}$. By the fact that $a^2 \geq 0$, we obtain

$$\begin{aligned}
[(a^2+3)-3]^2 &\geq 0 \\
(a^2+3)^2 - 4(a^2+3) + 4 &\geq 0 \\
(a^2+3)^2 &\geq 4(a^2+3) - 4 \\
(a^2+3)^2 &\geq 4(a^2+3-1) \\
(a^2+3)^2 &\geq 4(a^2+2) \\
\sqrt{(a^2+3)^2} &\geq 2\sqrt{a^2+2} \\
|a^2+3| &\geq 2\sqrt{a^2+2} \\
\frac{a^2+3}{\sqrt{a^2+2}} &\geq 2
\end{aligned}$$

□

3. **(10 marks)** Let $a \in \mathbb{R}$. Prove that

$$\frac{a^2+4}{\sqrt{a^2+3}} \geq 2.$$

Proof. Let $a \in \mathbb{R}$. By the fact that $a^2 \geq 0$, we obtain

$$\begin{aligned}
[(a^2+4)-4]^2 &\geq 0 \\
(a^2+4)^2 - 4(a^2+4) + 4 &\geq 0 \\
(a^2+4)^2 &\geq 4(a^2+4) - 4 \\
(a^2+4)^2 &\geq 4(a^2+4-1) \\
(a^2+4)^2 &\geq 4(a^2+3) \\
\sqrt{(a^2+4)^2} &\geq 2\sqrt{a^2+3} \\
|a^2+4| &\geq 2\sqrt{a^2+3} \\
\frac{a^2+4}{\sqrt{a^2+3}} &\geq 2
\end{aligned}$$

□

4. **(10 marks)** Let $a \in \mathbb{R}$. Prove that

$$\frac{a^2+5}{\sqrt{a^2+4}} \geq 2.$$

Proof. Let $a \in \mathbb{R}$. By the fact that $a^2 \geq 0$, we obtain

$$\begin{aligned}
[(a^2+5)-5]^2 &\geq 0 \\
(a^2+5)^2 - 4(a^2+5) + 4 &\geq 0 \\
(a^2+5)^2 &\geq 4(a^2+5) - 4 \\
(a^2+5)^2 &\geq 4(a^2+5-1) \\
(a^2+5)^2 &\geq 4(a^2+4) \\
\sqrt{(a^2+5)^2} &\geq 2\sqrt{a^2+4} \\
|a^2+5| &\geq 2\sqrt{a^2+4} \\
\frac{a^2+5}{\sqrt{a^2+4}} &\geq 2
\end{aligned}$$

□

No.2

1. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x+y| = |x-y|, \quad \text{then} \quad x|y| + y|x| = 0.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|x+y| = |x-y|$. Then

$$\begin{aligned} |x+y|^2 &= |x-y|^2 \\ (x+y)^2 &= (x-y)^2 \\ x^2 + 2xy + y^2 &= x^2 - 2xy + y^2 \\ 4xy &= 0 \\ xy &= 0 \end{aligned}$$

So, $x = 0$ or $y = 0$. It implies that $x|y| = 0$ and $|x|y = 0$. Thus, $x|y| + y|x| = 0$. □

2. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x+y| = |x+2y|, \quad \text{then} \quad |xy| = x^2.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|2x+y| = |x+2y|$. Then

$$\begin{aligned} |2x+y|^2 &= |x+2y|^2 \\ (2x+y)^2 &= (x+2y)^2 \\ 4x^2 + 4xy + y^2 &= x^2 + 4xy + 4y^2 \\ 3x^2 &= 3y^2 \\ x^2 &= y^2 \\ \sqrt{x^2} &= \sqrt{y^2} \\ |x| &= |y| \end{aligned}$$

It implies that

$$|xy| = |x||y| = |x||x| = |x|^2 = x^2.$$

□

3. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x - y| = |x - 2y|, \quad \text{then} \quad |xy| = x^2.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|2x - y| = |x - 2y|$. Then

$$\begin{aligned} |2x - y|^2 &= |x - 2y|^2 \\ (2x - y)^2 &= (x - 2y)^2 \\ 4x^2 - 4xy + y^2 &= x^2 - 4xy + 4y^2 \\ 3x^2 &= 3y^2 \\ x^2 &= y^2 \\ \sqrt{x^2} &= \sqrt{y^2} \\ |x| &= |y| \end{aligned}$$

It implies that

$$|xy| = |x||y| = |x||x| = |x|^2 = x^2.$$

□

4. **(10 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |2x + y| = |x + 2y|, \quad \text{then} \quad |xy| = x^2.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|2x + y| = |x + 2y|$. Then

$$\begin{aligned} |2x + y|^2 &= |x + 2y|^2 \\ (2x + y)^2 &= (x + 2y)^2 \\ 4x^2 + 4xy + y^2 &= x^2 + 4xy + 4y^2 \\ 3x^2 &= 3y^2 \\ x^2 &= y^2 \\ \sqrt{x^2} &= \sqrt{y^2} \\ |x| &= |y| \end{aligned}$$

It implies that

$$|xy| = |x||y| = |y||y| = |y|^2 = y^2.$$

□

No.3

1. **(10 marks)** Let $A = \left\{1 - \dfrac{n}{n^2+1} : n \in \mathbb{N}\right\}$.
What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution. Consider

$$A = \left\{\frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \ldots\right\}$$

Claim that $\inf A = \frac{1}{2}$ and $\sup A = 1$.

Proof. **We will prove that $\inf A = \frac{1}{2}$.**
Let $n \in \mathbb{N}$. By the fact that $(n-1)^2 \geq 0$,

$$\begin{aligned} n^2 - 2n + 1 &\geq 0 \\ n^2 + 1 &\geq 2n \\ \frac{1}{2} &\geq \frac{n}{n^2+1} \\ -\frac{1}{2} &\leq -\frac{n}{n^2+1} \\ 1 - \frac{1}{2} &\leq 1 - \frac{n}{n^2+1} \\ \frac{1}{2} &\leq 1 - \frac{n}{n^2+1} \end{aligned}$$

Thus, $\frac{1}{2}$ is a lower bound of A.
Let ℓ_0 be a lower bound of A. Then

$$\ell_0 \leq 1 - \frac{n}{n^2+1} \qquad \text{for all } n \in A$$

Since $1 \in \mathbb{N}$, $\frac{1}{2} = 1 - \frac{1}{1^2+1}$. Thus, $\ell_0 \leq \frac{1}{2}$. ☐

Proof. **We will prove that $\sup A = 1$.**
Let $n \in \mathbb{N}$. Then $\dfrac{n}{n^2+1} \geq 0$. So, $-\dfrac{n}{n^2+1} \leq 0$. Thus,

$$1 - \frac{n}{n^2+1} \leq 1.$$

So, 1 is an upper bound of A.
Let u be an upper bound of A such that $u < 1$. So, $1 - u > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < 1 - u$.
Since $n_0^2 + 1 \geq n_0^2$, $\dfrac{1}{n_0^2+1} \leq \dfrac{1}{n_0^2}$. We ontain

$$\frac{n_0}{n_0^2+1} \leq \frac{n_0}{n_0^2} = \frac{1}{n_0} < 1 - u$$

Thus,

$$u < 1 - \frac{n_0}{n_0^2+1}.$$

So, u is not an upper bound of A. It is contradiction. ☐

2. **(5 marks)** Let $A = \left\{1 - \dfrac{n}{n^2+2} : n \in \mathbb{N}\right\}$.
What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution. Consider

$$A = \left\{\frac{2}{3}, \frac{8}{11}, \frac{7}{9}, \ldots\right\}$$

Claim that $\inf A = \frac{2}{3}$ and $\sup A = 1$.

Proof. **We will prove that $\inf A = \frac{2}{3}$.**
Let $n \in \mathbb{N}$. Then $n - 1 \geq 0 > 1$. So, $(n-1)^2 \geq (n-1)$. We obtain

$$\begin{aligned} n^2 - 2n + 1 &\geq n - 1 \\ n^2 + 2 &\geq 3n \\ \frac{1}{3} &\geq \frac{n}{n^2+2} \\ -\frac{1}{3} &\leq -\frac{n}{n^2+2} \\ 1 - \frac{1}{3} &\leq 1 - \frac{n}{n^2+2} \\ \frac{2}{3} &\leq 1 - \frac{n}{n^2+2} \end{aligned}$$

Thus, $\frac{2}{3}$ is a lower bound of A.
Let ℓ_0 be a lower bound of A. Then

$$\ell_0 \leq 1 - \frac{n}{n^2+2} \qquad \text{for all } n \in A$$

Since $1 \in \mathbb{N}$, $\frac{1}{2} = 1 - \frac{1}{1^2+2}$. Thus, $\ell_0 \leq \frac{2}{3}$. □

Proof. **We will prove that $\sup A = 1$.**
Let $n \in \mathbb{N}$. Then $\dfrac{n}{n^2+2} \geq 0$. So, $-\dfrac{n}{n^2+2} \leq 0$. Thus,

$$1 - \frac{n}{n^2+2} \leq 1.$$

So, 1 is an upper bound of A.
Let u be an upper bound of A such that $u < 1$. So, $1 - u > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < 1 - u$.
Since $n_0^2 + 2 \geq n_0^2$, $\dfrac{1}{n_0^2+2} \leq \dfrac{1}{n_0^2}$. We ontain

$$\frac{n_0}{n_0^2+2} \leq \frac{n_0}{n_0^2} = \frac{1}{n_0} < 1 - u$$

Thus,

$$u < 1 - \frac{n_0}{n_0^2+2}.$$

So, u is not an upper bound of A. It is contradiction. □

3. **(5 marks)** Let $A = \left\{ 2 - \dfrac{n}{n^2+1} : n \in \mathbb{N} \right\}$.

What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution. Consider

$$A = \left\{ \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \ldots \right\}$$

Claim that $\inf A = \frac{3}{2}$ and $\sup A = 2$.

Proof. **We will prove that $\inf A = \frac{3}{2}$.**

Let $n \in \mathbb{N}$. By the fact that $(n-1)^2 \geq 0$,

$$\begin{aligned} n^2 - 2n + 1 &\geq 0 \\ n^2 + 1 &\geq 2n \\ \frac{1}{2} &\geq \frac{n}{n^2+1} \\ -\frac{1}{2} &\leq -\frac{n}{n^2+1} \\ 2 - \frac{1}{2} &\leq 2 - \frac{n}{n^2+1} \\ \frac{3}{2} &\leq 2 - \frac{n}{n^2+1} \end{aligned}$$

Thus, $\frac{3}{2}$ is a lower bound of A.

Let ℓ_0 be a lower bound of A. Then

$$\ell_0 \leq 2 - \frac{n}{n^2+1} \qquad \text{for all } n \in A$$

Since $1 \in \mathbb{N}$, $\frac{1}{2} = 2 - \frac{1}{1^2+1}$. Thus, $\ell_0 \leq \frac{3}{2}$. □

Proof. **We will prove that $\sup A = 2$.**

Let $n \in \mathbb{N}$. Then $\dfrac{n}{n^2+1} \geq 0$. So, $-\dfrac{n}{n^2+1} \leq 0$. Thus,

$$2 - \frac{n}{n^2+1} \leq 2.$$

So, 2 is an upper bound of A.

Let u be an upper bound of A such that $u < 2$. So, $2 - u > 0$.

By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < 2 - u$.

Since $n_0^2 + 1 \geq n_0^2$, $\dfrac{1}{n_0^2+1} \leq \dfrac{1}{n_0^2}$. We ontain

$$\frac{n_0}{n_0^2+1} \leq \frac{n_0}{n_0^2} = \frac{1}{n_0} < 2 - u$$

Thus,

$$u < 2 - \frac{n_0}{n_0^2+1}.$$

So, u is not an upper bound of A. It is contradiction. □

4. **(5 marks)** Let $A = \left\{ 2 - \dfrac{n}{n^2+2} : n \in \mathbb{N} \right\}$.
What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution. Consider

$$A = \left\{ \frac{5}{3}, \frac{19}{11}, \frac{16}{9}, ... \right\}$$

Claim that $\inf A = \frac{5}{3}$ and $\sup A = 2$.

Proof. **We will prove that $\inf A = \frac{5}{3}$.**
Let $n \in \mathbb{N}$. Then $n - 1 \geq 0 > 1$. So, $(n-1)^2 \geq (n-1)$. We obtain

$$\begin{aligned} n^2 - 2n + 1 &\geq n - 1 \\ n^2 + 2 &\geq 3n \\ \frac{1}{3} &\geq \frac{n}{n^2+2} \\ -\frac{1}{3} &\leq -\frac{n}{n^2+2} \\ 2 - \frac{1}{3} &\leq 2 - \frac{n}{n^2+2} \\ \frac{5}{3} &\leq 2 - \frac{n}{n^2+2} \end{aligned}$$

Thus, $\frac{5}{3}$ is a lower bound of A.
Let ℓ_0 be a lower bound of A. Then

$$\ell_0 \leq 2 - \frac{n}{n^2+2} \qquad \text{for all } n \in A$$

Since $1 \in \mathbb{N}$, $\frac{1}{2} = 2 - \frac{1}{1^2+2}$. Thus, $\ell_0 \leq \frac{5}{3}$. □

Proof. **We will prove that $\sup A = 2$.**
Let $n \in \mathbb{N}$. Then $\dfrac{n}{n^2+2} \geq 0$. So, $-\dfrac{n}{n^2+2} \leq 0$. Thus,

$$2 - \frac{n}{n^2+2} \leq 2.$$

So, 2 is an upper bound of A.
Let u be an upper bound of A such that $u < 2$. So, $2 - u > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < 2 - u$.
Since $n_0^2 + 2 \geq n_0^2$, $\dfrac{1}{n_0^2+2} \leq \dfrac{1}{n_0^2}$. We ontain

$$\frac{n_0}{n_0^2+2} \leq \frac{n_0}{n_0^2} = \frac{1}{n_0} < 2 - u$$

Thus,

$$u < 2 - \frac{n_0}{n_0^2+2}.$$

So, u is not an upper bound of A. It is contradiction. □

No.4

1. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+2)}{n^2+1} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$. Since $0 < 2n-1 < 2n$ and $n^2+1 > n^2$,

$$\frac{2n-1}{n^2+1} < \frac{2n-1}{n^2} < \frac{2n}{n^2} = \frac{2}{n}.$$

Hence,

$$\begin{aligned}\left|\frac{n(n+2)}{n^2+1} - 1\right| &= \left|\frac{(n^2+2n)-(n^2+1)}{n^2+1}\right| \\ &= \frac{2n-1}{n^2+1} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n(n+2)}{n^2+1} = 1$. □

2. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+3)}{n^2+2} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{3} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{3}$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$. Since $0 < 3n-2 < 3n$ and $n^2+2 > n^2$,

$$\frac{3n-2}{n^2+2} < \frac{3n-2}{n^2} < \frac{3n}{n^2} = \frac{3}{n}.$$

Hence,

$$\begin{aligned}\left|\frac{n(n+3)}{n^2+2} - 1\right| &= \left|\frac{(n^2+3n)-(n^2+2)}{n^2+2}\right| \\ &= \frac{3n-2}{n^2+2} < \frac{3}{n} \leq \frac{3}{N} < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n(n+3)}{n^2+2} = 1$. □

3. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+3)}{n^2+1} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\dfrac{\varepsilon}{3} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{3}$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\dfrac{1}{n} \leq \dfrac{1}{N}$. Since $0 < 3n - 1 < 3n$ and $n^2 + 1 > n^2$,

$$\frac{3n-1}{n^2+1} < \frac{3n-1}{n^2} < \frac{3n}{n^2} = \frac{3}{n}.$$

Hence,

$$\begin{aligned}\left|\frac{n(n+3)}{n^2+1} - 1\right| &= \left|\frac{(n^2+3n)-(n^2+1)}{n^2+1}\right| \\ &= \frac{3n-1}{n^2+1} < \frac{3}{n} \leq \frac{3}{N} < \varepsilon.\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{n(n+3)}{n^2+1} = 1.$ □

4. **(10 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n(n+5)}{n^2+2} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\dfrac{\varepsilon}{2} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{2}$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\dfrac{1}{n} \leq \dfrac{1}{N}$. Since $0 < 5n - 2 < 5n$ and $n^2 + 2 > n^2$,

$$\frac{5n-2}{n^2+2} < \frac{5n-2}{n^2} < \frac{5n}{n^2} = \frac{5}{n}.$$

Hence,

$$\begin{aligned}\left|\frac{n(n+5)}{n^2+2} - 1\right| &= \left|\frac{(n^2+5n)-(n^2+2)}{n^2+2}\right| \\ &= \frac{5n-2}{n^2+2} < \frac{5}{n} \leq \frac{5}{N} < \varepsilon.\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{n(n+5)}{n^2+2} = 1.$ □

No.5

1. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+1}) = 0.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$.
Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ sucth that

$$\text{for all } n \geq N, \text{ it implies that } |x_n - a| < \frac{\varepsilon}{2}. \qquad ... \; (*)$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n+1 > n \geq N$. So, n and $n+1$ satisfy $(*)$. We obtain

$$\begin{aligned} |x_n - x_{n+1} - 0| &= |(x_n - a) - (x_{n+1} - a)| \\ &\leq |x_n - a| + |x_{n+1} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

2. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+2}) = 0.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$.
Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ sucth that

$$\text{for all } n \geq N, \text{ it implies that } |x_n - a| < \frac{\varepsilon}{2}. \qquad ... \; (*)$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n+2 > n \geq N$. So, n and $n+2$ satisfy $(*)$. We obtain

$$\begin{aligned} |x_n - x_{n+2} - 0| &= |(x_n - a) - (x_{n+2} - a)| \\ &\leq |x_n - a| + |x_{n+2} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

3. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+3}) = 0.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$.
Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ sucth that

$$\text{for all } n \geq N, \text{ it implies that } |x_n - a| < \frac{\varepsilon}{2}. \qquad \text{... } (*)$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n + 3 > n \geq N$. So, n and $n + 3$ satisfy $(*)$. We obtain

$$\begin{aligned} |x_n - x_{n+3} - 0| &= |(x_n - a) - (x_{n+3} - a)| \\ &\leq |x_n - a| + |x_{n+3} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

4. **(10 marks)** Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}(x_n - x_{n+4}) = 0.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$.
Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ sucth that

$$\text{for all } n \geq N, \text{ it implies that } |x_n - a| < \frac{\varepsilon}{2}. \qquad \text{... } (*)$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n + 4 > n \geq N$. So, n and $n + 4$ satisfy $(*)$. We obtain

$$\begin{aligned} |x_n - x_{n+4} - 0| &= |(x_n - a) - (x_{n+4} - a)| \\ &\leq |x_n - a| + |x_{n+4} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

No.6

1. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}. \qquad \text{... } (*)$$

Let $\varepsilon > 0$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $(*)$ holds, i.e.,

$$|x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N} \qquad \text{... } (**)$$

Since $\varepsilon > 0$, by Archimedean property, there is $d \in \mathbb{N}$ such that $\dfrac{1}{d} < \varepsilon$. From $(**)$,

$$|x_n - x_m| < \frac{1}{d} < \varepsilon.$$

because $d \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. We conclude that $\{a_n\}$ converges. $\square$

2. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}. \qquad \text{... } (*)$$

Let $\varepsilon > 0$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $(*)$ holds, i.e.,

$$|x_n - x_m| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N} \qquad \text{... } (**)$$

Since $\sqrt{\varepsilon} > 0$, by Archimedean property, there is $d \in \mathbb{N}$ such that $\frac{1}{d} < \sqrt{\varepsilon}$. Then $\frac{1}{d^2} < \varepsilon$.
From $(**)$,

$$|x_n - x_m| < \frac{1}{d^2} < \varepsilon.$$

because $d \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. We conclude that $\{a_n\}$ converges. □

3. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^3} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^3} \quad \text{for all } k \in \mathbb{N}. \qquad ... \ (*)$$

Let $\varepsilon > 0$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $(*)$ holds, i.e.,

$$|x_n - x_m| < \frac{1}{k^3} \quad \text{for all } k \in \mathbb{N} \qquad ... \ (**)$$

Since $\sqrt[3]{\varepsilon} > 0$, by Archimedean property, there is $d \in \mathbb{N}$ such that $\frac{1}{d} < \sqrt[3]{\varepsilon}$. Then $\frac{1}{d^3} < \varepsilon$.
From $(**)$,

$$|x_n - x_m| < \frac{1}{d^3} < \varepsilon.$$

because $d \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. We conclude that $\{a_n\}$ converges. $\square$

4. **(10 marks)** Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^4} \quad \text{for all } k \in \mathbb{N}.$$

Prove that $\{a_n\}$ converges.
Hint: Show that $\{a_n\}$ is Cauchy.

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement :

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k^4} \quad \text{for all } k \in \mathbb{N}. \qquad \text{... } (*)$$

Let $\varepsilon > 0$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $(*)$ holds, i.e.,

$$|x_n - x_m| < \frac{1}{k^4} \quad \text{for all } k \in \mathbb{N} \qquad \text{... } (**)$$

Since $\sqrt[4]{\varepsilon} > 0$, by Archimedean property, there is $d \in \mathbb{N}$ such that $\frac{1}{d} < \sqrt[4]{\varepsilon}$. Then $\frac{1}{d^4} < \varepsilon$.
From $(**)$,

$$|x_n - x_m| < \frac{1}{d^4} < \varepsilon.$$

because $d \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. We conclude that $\{a_n\}$ converges. ☐

No.7

1. **(10 marks)** Let F be a closed set. Assume that $\{x_n\}$ is a sequence in F (it means that $x_n \in F$ for all $n \in \mathbb{N}$). Prove that

$$\text{if} \quad x_n \to a \text{ as } n \to \infty, \quad \text{then} \quad a \in F.$$

Proof. Let F be a closed set. Assume that $\{x_n\}$ is a sequence in F. We will prove by contradiction.
Assume that $x_n \to a$ as $n \to \infty$ and $a \notin F$. Then $a \in F^c$.
Since F^c is open, there $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq F^c$. So,

$$(a - \delta, a + \delta) \cap F = \varnothing. \qquad ... \ (*)$$

From $x_n \to a$ as $n \to \infty$, $(\varepsilon = \delta)$ there is an $N \in \mathbb{N}$ such that $n \geq N$

$$|x_n - a| < \delta.$$

Then $x_n \in (a - \delta, a + \delta)$. But $x_n \in F$, this is contradiction to $(*)$.
Thus, $a \in F$. □

No.8

1. **(10 marks)** Use definition to prove that

$$\lim_{x\to 1}\left(x+\frac{1}{x}\right)=2.$$

Proof. Let $\varepsilon>0$. Choose $\delta=\min\left\{0.5,\sqrt{\frac{\varepsilon}{2}}\right\}$. Suppose that $0<|x-1|<\delta$. Then $0<|x-1|<0.5$,

$$0.5<x<1 \quad \text{or} \quad 1<x<1.5. \text{ So,}$$

$$0.5<|x|<1.$$

We obtain $\frac{1}{|x|}<2$. Then,

$$\begin{aligned}\left|x+\frac{1}{x}-2\right| &= \left|\frac{x^2-2x+1}{x}\right| = \left|\frac{(x-1)^2}{x}\right| \\ &= \frac{1}{|x|}\cdot|x-1|^2 \\ &< 2\cdot\delta^2 < 2\cdot\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore, $\lim_{x\to 1}\left(x+\frac{1}{x}\right)=2$. □

2. **(10 marks)** Use definition to prove that

$$\lim_{x\to -1}\left(x+\frac{1}{x}\right)=0.$$

Proof. Let $\varepsilon>0$. Choose $\delta=\min\left\{0.5,\frac{\varepsilon}{2}\right\}$. Suppose that $0<|x+1|<\delta$. Then $0<|x+1|<0.5$,

$$-1.5<x<-1 \quad \text{or} \quad -1<x<-0.5. \text{ So,}$$

$$0.5<|x|<1.5.$$

We obtain $\frac{1}{|x|}<2$. Then,

$$\begin{aligned}\left|x+\frac{1}{x}-0\right| &= \left|\frac{x+1}{x}\right| = \frac{1}{|x|}\cdot|x+1| \\ &< 2\cdot\delta < 2\cdot\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore, $\lim_{x\to -1}\left(x+\frac{1}{x}\right)=0$. □

3. **(10 marks)** Use definition to prove that

$$\lim_{x\to 2}\left(x+\frac{2}{x}\right)=3.$$

Proof. Let $\varepsilon>0$. Choose $\delta=\min\left\{1,\frac{\varepsilon}{4}\right\}$. Suppose that $0<|x-2|<\delta$. Then $0<|x-2|<1$,

$$1<x<2 \quad \text{or} \quad 2<x<3. \text{ So,}$$

$$1<|x|<3.$$

We obtain $|x|<3$ and $\frac{1}{|x|}<1$. Then,

$$\begin{aligned}\left|x+\frac{2}{x}-3\right| &= \left|\frac{x^2-3x+2}{x}\right| = \left|\frac{(x-1)(x-2)}{x}\right| \\ &= \frac{1}{|x|}\cdot|x-1||x-2| \\ &= 1(|x|+1)\delta<(3+1)\delta \\ &< 4\cdot\frac{\varepsilon}{4}=\varepsilon.\end{aligned}$$

Therefore, $\lim_{x\to 2}\left(x+\frac{2}{x}\right)=3.$ □

4. **(10 marks)** Use definition to prove that

$$\lim_{x\to -2}\left(x+\frac{2}{x}\right)=-3.$$

Proof. Let $\varepsilon>0$. Choose $\delta=\min\left\{1,\frac{\varepsilon}{4}\right\}$. Suppose that $0<|x-2|<\delta$. Then $0<|x+2|<1$,

$$-3<x<-2 \quad \text{or} \quad -2<x<-1. \text{ So,}$$

$$1<|x|<3.$$

We obtain $|x|<3$ and $\frac{1}{|x|}<1$. Then,

$$\begin{aligned}\left|x+\frac{2}{x}+3\right| &= \left|\frac{x^2+3x+2}{x}\right| = \left|\frac{(x+1)(x+2)}{x}\right| \\ &= \frac{1}{|x|}\cdot|x+1||x+2| \\ &= 1(|x|+1)\delta<(3+1)\delta \\ &< 4\cdot\frac{\varepsilon}{4}=\varepsilon.\end{aligned}$$

Therefore, $\lim_{x\to -2}\left(x+\frac{2}{x}\right)=-3.$ □

No.9

1. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x \to 1^+} (x + f(x)) = +\infty.$$

Prove that $f(x) \to +\infty$ as $x \to 1^+$.

Proof. Let f be a real value function. Assume that

$$\lim_{x \to 1^+} (x + f(x)) = +\infty.$$

Let $M > 0$. There is a $\delta_1 > 0$ such that $0 < x - 1 < \delta_1$. It implies that

$$x + f(x) > M + 2. \qquad (*)$$

Chose $\delta = \min\{1, \delta_1\}$. Let $x \in \mathbb{R}$ such that $0 < x - 1 < \delta$.
Then $0 < x - 1 < 1$ or $1 < x < 2$. So, $-x > -2$ and x satisfies $(*)$.
We obtain

$$f(x) = (x + f(x)) - x > (M + 2) - 2 = M.$$

Thus, $f(x) \to +\infty$ as $x \to 1^+$. $\square$

2. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x \to 1^-} (x + f(x)) = +\infty.$$

Prove that $f(x) \to +\infty$ as $x \to 1^-$.

Proof. Let f be a real value function. Assume that

$$\lim_{x \to 1^-} (x + f(x)) = +\infty.$$

Let $M > 0$. There is a $\delta_1 > 0$ such that $-\delta_1 < x - 1 < 0$. It implies that

$$x + f(x) > M + 1. \qquad (*)$$

Chose $\delta = \min\{1, \delta_1\}$. Let $x \in \mathbb{R}$ such that $-\delta < x - 1 < 0$.
Then $-1 < x - 1 < 0$ or $0 < x < 1$. So, $-x > -1$ and x satisfies $(*)$.
We obtain

$$f(x) = (x + f(x)) - x > (M + 1) - 1 = M.$$

Thus, $f(x) \to +\infty$ as $x \to 1^-$. $\square$

3. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x \to 1^+} (x + f(x)) = -\infty.$$

Prove that $f(x) \to -\infty$ as $x \to 1^+$.

Proof. Let f be a real value function. Assume that

$$\lim_{x \to 1^+} (x + f(x)) = -\infty.$$

Let $M < 0$. There is a $\delta_1 > 0$ such that $0 < x - 1 < \delta_1$. It implies that

$$x + f(x) < M. \qquad (*)$$

Chose $\delta = \min\{1, \delta_1\}$. Let $x \in \mathbb{R}$ such that $0 < x - 1 < \delta$.
Then $0 < x - 1 < 1$ or $1 < x < 2$. So, $-x < -1$ and x satisfies $(*)$.
We obtain

$$f(x) = (x + f(x)) - x < M - 1 < M.$$

Thus, $f(x) \to -\infty$ as $x \to 1^+$. □

4. **(10 marks)** Let f be a real value function. Assume that

$$\lim_{x \to 1^-} (x + f(x)) = -\infty.$$

Prove that $f(x) \to -\infty$ as $x \to 1^-$.

Proof. Let f be a real value function. Assume that

$$\lim_{x \to 1^-} (x + f(x)) = -\infty.$$

Let $M < 0$. There is a $\delta_1 > 0$ such that $-\delta_1 < x - 1 < 0$. It implies that

$$x + f(x) < M. \qquad (*)$$

Chose $\delta = \min\{1, \delta_1\}$. Let $x \in \mathbb{R}$ such that $-\delta < x - 1 < 0$.
Then $-1 < x - 1 < 0$ or $0 < x < 1$. So, $-x < 0$ and x satisfies $(*)$.
We obtain

$$f(x) = (x + f(x)) - x < M + 0 = M.$$

Thus, $f(x) \to -\infty$ as $x \to 1^-$. □

No.10

1. **(10 marks)** Use definition (sequence) to prove that

$$\lim_{n\to\infty}\left(1-\sqrt{n}\right) = -\infty.$$

Proof. Let $M \in \mathbb{R}$.
Case $M \geq 1$. It is easy to see that

$$1-\sqrt{n} \leq 0 < 1 \leq M \quad \text{for all } n \in \mathbb{N}.$$

Case $M < 1$. Then $1 - M > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $(1-M)^2 < N$. It is equivalent to

$$1-\sqrt{N} < M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\sqrt{n} \geq \sqrt{N}$. So, $-\sqrt{n} \leq -\sqrt{N}$. We obtain

$$1-\sqrt{n} \leq 1-\sqrt{N} < M.$$

□

2. **(10 marks)** Use definition (sequence) to prove that

$$\lim_{n\to\infty}\left(\sqrt{n}-1\right) = \infty.$$

Proof. Let $M \in \mathbb{R}$.
Case $M \leq -1$. It is easy to see that

$$\sqrt{n}-1 \geq 0 > -1 \geq M \quad \text{for all } n \in \mathbb{N}.$$

Case $M > -1$. Then $M + 1 > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $(1+M)^2 < N$. It is equivalent to

$$\sqrt{N}-1 > M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\sqrt{n} \geq \sqrt{N}$. We obtain

$$\sqrt{n}-1 \geq \sqrt{N}-1 > M.$$

□

3. **(10 marks)** Use definition (function) to prove that

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0.$$

Proof. Let $\varepsilon>0$. Choose $M=\dfrac{1}{\varepsilon}$. Then $M>0$.
Let $x\in\mathbb{R}$ such that $x>M>0$. It follows that $\dfrac{1}{x}<\dfrac{1}{M}$. We obtain

$$\begin{aligned}\left|\sqrt{x^2+1}-x-0\right| &= \left|(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right| \\ &= \left|\frac{1}{\sqrt{x^2+1}+x}\right| \\ &< \frac{1}{x} && \because \sqrt{x^2+1}+x>x \\ &< \frac{1}{M}=\varepsilon\end{aligned}$$

Thus, $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$. □

4. **(10 marks)** Use definition (function) to prove that

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1}+x\right)=0.$$

Proof. Let $\varepsilon>0$. Choose $M=-\dfrac{1}{\varepsilon}$. Then $M<0$.
Let $x\in\mathbb{R}$ such that $x<M<0$. Then $-x>-M>0$. It follows that $\dfrac{1}{-x}<\dfrac{1}{-M}$. We obtain

$$\begin{aligned}\left|\sqrt{x^2+1}+x-0\right| &= \left|(\sqrt{x^2+1}+x)\cdot\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}\right| \\ &= \left|\frac{1}{\sqrt{x^2+1}-x}\right| \\ &< \frac{1}{-x} && \because \sqrt{x^2+1}-x>-x \\ &< \frac{1}{-M}=\varepsilon\end{aligned}$$

Thus, $\lim_{x\to-\infty}\left(\sqrt{x^2+1}+x\right)=0$. □