QUIZ 1 : MAC3309 Mathematical Analysis

Topic Field axioms and Completeness axioms **Score** 10 marks
Time Wendsday 15 Dec 2021, 3rd Week, Semester 2/2021
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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No.1

1. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x+y| = |x| + |y|, \quad \text{then} \quad xy \geq 0.$$

2. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x-y| = |x| + |y|, \quad \text{then} \quad xy \leq 0.$$

No.2

1. **(5 marks)** Let $A = \left\{1 + \dfrac{1}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

2. **(5 marks)** Let $A = \left\{2 + \dfrac{1}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

3. **(5 marks)** Let $A = \left\{3 + \dfrac{1}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

4. **(5 marks)** Let $A = \left\{1 + \dfrac{2}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

5. **(5 marks)** Let $A = \left\{1 + \dfrac{3}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution QUIZ 1 : MAC3309 Mathematical Analysis

Topic	Field axioms and Completeness axioms **Score** 10 marks
Time	Wendsday 15 Dec 2021, 3rd Week, Semester 2/2021
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No.1

1. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x+y| = |x| + |y|, \quad \text{then} \quad xy \geq 0.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|x+y| = |x| + |y|$. Then

$$\begin{aligned} |x+y|^2 &= (|x|+|y|)^2 \\ (x+y)^2 &= |x|^2 + 2|x||y| + |y|^2 \\ x^2 + 2xy + y^2 &= x^2 + 2|xy| + y^2 \\ xy &= |xy| \end{aligned}$$

By definition of absolute value, it implies that $xy \geq 0$. □

2. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that

$$\text{if} \quad |x-y| = |x| + |y|, \quad \text{then} \quad xy \leq 0.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $|x-y| = |x| + |y|$. Then

$$\begin{aligned} |x-y|^2 &= (|x|+|y|)^2 \\ (x-y)^2 &= |x|^2 + 2|x||y| + |y|^2 \\ x^2 - 2xy + y^2 &= x^2 + 2|xy| + y^2 \\ -xy &= |xy| \end{aligned}$$

By definition of absolute value, it implies that $xy \leq 0$. □

No.2

1. **(5 marks)** Let $A = \left\{1 + \frac{1}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ?
   Verify (proof) your answers.

   Solution. Consider

   $$A = \left\{2, \frac{5}{4}, \frac{10}{9}, \frac{17}{16}, \dots\right\}$$

   Claim that $\inf A = 1$ and $\sup A = 2$.

   Proof. **We will prove that $\inf A = 1$.**
   Let $n \in \mathbb{N}$. Then $n^2 \geq 0$. So, $\frac{1}{n^2} \geq 0$. Thus,

   $$1 \leq 1 + \frac{1}{n^2}$$

   Thus, 1 is a lower bound of A.
   Let ℓ_0 be a lower bound of A such that $1 < \ell_0$. Then $\sqrt{\ell_0 - 1} > 0$.
   By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

   $$\frac{1}{n_0} < \sqrt{\ell_0 - 1}$$
   $$\frac{1}{n_0^2} < \ell_0 - 1$$
   $$1 + \frac{1}{n_0^2} < \ell_0$$

   So, ℓ_0 is not a lower bound of A. It is contradiction. □

   Proof. **We will prove that $\sup A = 2$.**
   Let $n \in \mathbb{N}$. Then $n^2 \geq 1$. So, $\frac{1}{n^2} \leq 1$. Thus,

   $$1 + \frac{1}{n^2} \leq 2$$

   Thus, 2 is an upper bound of A.
   Let u be an upper bound of A. Then

   $$1 + \frac{1}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}$$

   Since $1 \in \mathbb{N}$, $2 = 1 + \frac{1}{1^2} \in A$. Thus, $2 \leq u$. □

2. **(5 marks)** Let $A = \left\{ 2 + \dfrac{1}{n^2} : n \in \mathbb{N} \right\}$. What are **supremum** and **infimum** of A ?
Verify (proof) your answers.

Solution. Consider

$$A = \left\{ 3, \frac{9}{4}, \frac{19}{9}, \frac{33}{16}, \dots \right\}$$

Claim that $\inf A = 2$ and $\sup A = 3$.

Proof. **We will prove that $\inf A = 2$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 0$. So, $\dfrac{1}{n^2} \geq 0$. Thus,

$$2 \leq 2 + \frac{1}{n^2}$$

Thus, 2 is a lower bound of A.
Let ℓ_0 be a lower bound of A such that $2 < \ell_0$. Then $\sqrt{\ell_0 - 2} > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{n_0} &< \sqrt{\ell_0 - 2} \\ \frac{1}{n_0^2} &< \ell_0 - 2 \\ 2 + \frac{1}{n_0^2} &< \ell_0 \end{aligned}$$

So, ℓ_0 is not a lower bound of A. It is contradiction. □

Proof. **We will prove that $\sup A = 3$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 1$. So, $\dfrac{1}{n^2} \leq 1$. Thus,

$$2 + \frac{1}{n^2} \leq 3$$

Thus, 3 is an upper bound of A.
Let u be an upper bound of A. Then

$$2 + \frac{1}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}$$

Since $1 \in \mathbb{N}$, $3 = 2 + \dfrac{1}{1^2} \in A$. Thus, $3 \leq u$. □

3. **(5 marks)** Let $A = \left\{ 3 + \frac{1}{n^2} : n \in \mathbb{N} \right\}$. What are **supremum** and **infimum** of A ?
Verify (proof) your answers.

Solution. Consider

$$A = \left\{ 4, \frac{13}{4}, \frac{28}{9}, \frac{49}{16}, \dots \right\}$$

Claim that $\inf A = 3$ and $\sup A = 4$.

Proof. **We will prove that $\inf A = 3$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 0$. So, $\frac{1}{n^2} \geq 0$. Thus,

$$3 \leq 3 + \frac{1}{n^2}$$

Thus, 3 is a lower bound of A.
Let ℓ_0 be a lower bound of A such that $3 < \ell_0$. Then $\sqrt{\ell_0 - 3} > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{n_0} &< \sqrt{\ell_0 - 3} \\ \frac{1}{n_0^2} &< \ell_0 - 3 \\ 3 + \frac{1}{n_0^2} &< \ell_0 \end{aligned}$$

So, ℓ_0 is not a lower bound of A. It is contradiction. □

Proof. **We will prove that $\sup A = 4$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 1$. So, $\frac{1}{n^2} \leq 1$. Thus,

$$3 + \frac{1}{n^2} \leq 4$$

Thus, 4 is an upper bound of A.
Let u be an upper bound of A. Then

$$3 + \frac{1}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}$$

Since $1 \in \mathbb{N}$, $4 = 3 + \frac{1}{1^2} \in A$. Thus, $4 \leq u$. □

4. **(5 marks)** Let $A = \left\{1 + \frac{2}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ? Verify (proof) your answers.

Solution. Consider

$$A = \left\{3, \frac{3}{2}, \frac{11}{9}, \frac{9}{8}, \ldots\right\}$$

Claim that $\inf A = 1$ and $\sup A = 3$.

Proof. **We will prove that $\inf A = 1$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 0$. So, $\frac{2}{n^2} \geq 0$. Thus,

$$1 \leq 1 + \frac{2}{n^2}$$

Thus, 1 is a lower bound of A.
Let ℓ_0 be a lower bound of A such that $1 < \ell_0$. Then $\sqrt{\frac{\ell_0 - 1}{2}} > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{n_0} &< \sqrt{\frac{\ell_0 - 1}{2}} \\ \frac{1}{n_0^2} &< \frac{\ell_0 - 1}{2} \\ \frac{2}{n_0^2} &< \ell_0 - 1 \\ 1 + \frac{2}{n_0^2} &< \ell_0 \end{aligned}$$

So, ℓ_0 is not a lower bound of A. It is contradiction. □

Proof. **We will prove that $\sup A = 3$.**
Let $n \in \mathbb{N}$. Then $n^2 \geq 1$. So, $\frac{1}{n^2} \leq 1$. Thus,

$$\begin{aligned} \frac{2}{n^2} &\leq 2 \\ 1 + \frac{2}{n^2} &\leq 3 \end{aligned}$$

Thus, 3 is an upper bound of A.
Let u be an upper bound of A. Then

$$1 + \frac{2}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}$$

Since $1 \in \mathbb{N}$, $3 = 1 + \frac{2}{1^2} \in A$. Thus, $3 \leq u$. □

5. **(5 marks)** Let $A = \left\{1 + \frac{3}{n^2} : n \in \mathbb{N}\right\}$. What are **supremum** and **infimum** of A ?

Verify (proof) your answers.

Solution. Consider

$$A = \left\{4, \frac{7}{4}, \frac{4}{3}, \frac{19}{16}, \ldots\right\}$$

Claim that $\inf A = 1$ and $\sup A = 4$.

Proof. **We will prove that $\inf A = 1$.**

Let $n \in \mathbb{N}$. Then $n^2 \geq 0$. So, $\frac{3}{n^2} \geq 0$. Thus,

$$1 \leq 1 + \frac{3}{n^2}$$

Thus, 1 is a lower bound of A.

Let ℓ_0 be a lower bound of A such that $1 < \ell_0$. Then $\sqrt{\frac{\ell_0 - 1}{3}} > 0$.

By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{n_0} &< \sqrt{\frac{\ell_0 - 1}{3}} \\ \frac{1}{n_0^2} &< \frac{\ell_0 - 1}{3} \\ \frac{3}{n_0^2} &< \ell_0 - 1 \\ 1 + \frac{3}{n_0^2} &< \ell_0 \end{aligned}$$

So, ℓ_0 is not a lower bound of A. It is contradiction. □

Proof. **We will prove that $\sup A = 4$.**

Let $n \in \mathbb{N}$. Then $n^2 \geq 1$. So, $\frac{1}{n^2} \leq 1$. Thus,

$$\begin{aligned} \frac{3}{n^2} &\leq 3 \\ 1 + \frac{3}{n^2} &\leq 4 \end{aligned}$$

Thus, 4 is an upper bound of A.

Let u be an upper bound of A. Then

$$1 + \frac{3}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}$$

Since $1 \in \mathbb{N}$, $4 = 1 + \frac{3}{1^2} \in A$. Thus, $4 \leq u$. □

QUIZ 2 : MAC3309 Mathematical Analysis

Topic Limit of Sequences **Score** 10 marks
Time Thurday 6 Jan 2022, 5rd Week, Semester 2/2021
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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No.1

1. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} \quad \text{exists.}$$

2. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+2} \quad \text{exists.}$$

3. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+3} \quad \text{exists.}$$

4. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+4} \quad \text{exists.}$$

5. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+5} \quad \text{exists.}$$

No.2

1. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt{n+1} = +\infty.$$

2. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt{n+2} = +\infty.$$

3. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt[3]{2-n} = -\infty.$$

4. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt[3]{1-n} = -\infty.$$

Solution QUIZ 2 : MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	Thurday 6 Jan 2022, 5rd Week, Semester 2/2021		
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University		

No.1

1. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{2}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{2}}$. It is equivalent to

$$\frac{2}{N^2} < \varepsilon.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\frac{2}{n^2} \leq \frac{2}{N^2}$. Since $n^2+1 > n^2$, $\frac{2}{n^2+1} < \frac{2}{n^2}$. Hence,

$$\begin{aligned} \left| \frac{n^2-1}{n^2+1} - 1 \right| &= \left| \frac{(n^2-1)-(n^2+1)}{n^2+1} \right| \\ &= \frac{2}{n^2+1} < \frac{2}{n^2} \leq \frac{2}{N^2} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$. □

2. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+2} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{4}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{4}}$. It is equivalent to

$$\frac{4}{N^2} < \varepsilon.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\frac{4}{n^2} \leq \frac{4}{N^2}$. Since $n^2+2 > n^2$, $\frac{4}{n^2+2} < \frac{4}{n^2}$. Hence,

$$\begin{aligned} \left| \frac{n^2-2}{n^2+2} - 1 \right| &= \left| \frac{(n^2-2)-(n^2+2)}{n^2+2} \right| \\ &= \frac{4}{n^2+2} < \frac{4}{n^2} \leq \frac{4}{N^2} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^2-2}{n^2+2} = 1$. □

3. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+3} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{4}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{4}}$. It is equivalent to

$$\frac{4}{N^2} < \varepsilon.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\frac{4}{n^2} \leq \frac{4}{N^2}$. Since $n^2+3 > n^2$, $\frac{4}{n^2+3} < \frac{4}{n^2}$. Hence,

$$\begin{aligned} \left|\frac{n^2-1}{n^2+3} - 1\right| &= \left|\frac{(n^2-1)-(n^2+3)}{n^2+3}\right| \\ &= \frac{4}{n^2+3} < \frac{4}{n^2} \leq \frac{4}{N^2} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^2-1}{n^2+3} = 1$. □

4. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+4} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{6}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{6}}$. It is equivalent to

$$\frac{6}{N^2} < \varepsilon.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\frac{6}{n^2} \leq \frac{6}{N^2}$. Since $n^2+4 > n^2$, $\frac{6}{n^2+4} < \frac{6}{n^2}$. Hence,

$$\begin{aligned} \left|\frac{n^2-2}{n^2+4} - 1\right| &= \left|\frac{(n^2-2)-(n^2+4)}{n^2+4}\right| \\ &= \frac{6}{n^2+4} < \frac{6}{n^2} \leq \frac{6}{N^2} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^2-2}{n^2+4} = 1$. □

5. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+5} \quad \text{exists.}$$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{6}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{6}}$. It is equivalent to

$$\frac{6}{N^2} < \varepsilon.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\frac{6}{n^2} \leq \frac{6}{N^2}$. Since $n^2+5 > n^2$, $\frac{6}{n^2+5} < \frac{6}{n^2}$. Hence,

$$\begin{aligned} \left|\frac{n^2-1}{n^2+5} - 1\right| &= \left|\frac{(n^2-1)-(n^2+5)}{n^2+5}\right| \\ &= \frac{6}{n^2+5} < \frac{6}{n^2} \leq \frac{6}{N^2} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^2-1}{n^2+5} = 1$. □

No.2

1. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt{n+1} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $M^2 - 1 < N$. It is equivalent to

$$\sqrt{N+1} > M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n + 1 \geq N + 1$. So, $\sqrt{n+1} \geq \sqrt{N+1}$. We obtain

$$\sqrt{n+1} \geq \sqrt{N+1} > M.$$

□

2. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt{n+2} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $M^2 - 2 < N$. It is equivalent to

$$\sqrt{N+2} > M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n + 2 \geq N + 2$. So, $\sqrt{n+2} > \sqrt{N+2}$. We obtain

$$\sqrt{n+2} \geq \sqrt{N+2} > M.$$

□

3. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt[3]{2-n} = -\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $2 - M^3 < N$. It is equivalent to

$$\sqrt[3]{2-N} < M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $-n \leq -N$. So, $2 - n \leq 2 - N$. We obtain

$$\sqrt[3]{2-n} \leq \sqrt[3]{2-N} < M.$$

□

4. **(5 marks)** Use definition to prove that

$$\lim_{n\to\infty} \sqrt[3]{1-n} = -\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $1 - M^3 < N$. It is equivalent to

$$\sqrt[3]{1-N} < M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $-n \leq -N$. So, $1 - n \leq 1 - N$. We obtain

$$\sqrt[3]{1-n} \leq \sqrt[3]{1-N} < M.$$

□

QUIZ 3-4 : MAC3309 Mathematical Analysis

Topic Continuity and Riemann Sum **Score** 20 marks
Time Thurday 10 Mar. 2022, 15rd Week, Semester 2/2021
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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No.1

1. Use definition to prove that $f(x) = \dfrac{1}{x^2}$ is continuous at $x = 1$.
2. Use definition to prove that $f(x) = \dfrac{1}{x^2}$ is continuousat $x = -1$.
3. Use definition to prove that $f(x) = \dfrac{2}{x^2}$ is continuous at $x = 1$.
4. Use definition to prove that $f(x) = \dfrac{2}{x^2}$ is continuous at $x = -1$.
5. Use definition to prove that $f(x) = \dfrac{1}{2x^2}$ is continuous at $x = 1$.
6. Use definition to prove that $f(x) = \dfrac{1}{2x^2}$ is continuous $x = -1$.

No.2

1. Let $f(x) = x^2 - 1$ where $x \in [1, 2]$ and $P = \left\{1 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1, 2]$. Find $I(f)$.
2. Let $f(x) = x^2 + 1$ where $x \in [1, 2]$ and $P = \left\{1 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1, 2]$. Find $I(f)$.
3. Let $f(x) = x^2 - 2$ where $x \in [1, 2]$ and $P = \left\{1 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1, 2]$. Find $I(f)$.
4. Let $f(x) = x^2 - 1$ where $x \in [2, 3]$ and $P = \left\{2 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2, 3]$. Find $I(f)$.
5. Let $f(x) = x^2 + 1$ where $x \in [2, 3]$ and $P = \left\{2 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2, 3]$. Find $I(f)$.
6. Let $f(x) = x^2 - 2$ where $x \in [2, 3]$ and $P = \left\{2 + \dfrac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2, 3]$. Find $I(f)$.

Solution QUIZ 3-4 : MAC3309 Mathematical Analysis

Topic Continuity and Riemann Sum **Score** 20 marks
Time Thurday 10 Mar. 2022, 15rd Week, Semester 2/2021
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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No.1

1. Use definition to prove that $f(x) = \dfrac{1}{x^2}$ is continuous at $x = 1$.

   Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{10}\}$ such that $|x-1| < \delta$. Then $|x-1| < 0.5$. So,

   $$-0.5 < x - 1 < 0.5.$$

   We obtain

   $$0.5 < x < 1.5 \text{ and } 1.5 < x + 1 < 2.5.$$

   Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x+1| < 2.5$. We obtain

   $$\begin{aligned} |f(x) - f(1)| &= \left|\frac{1}{x^2} - 1\right| = \left|\frac{1-x^2}{x^2}\right| = \left|\frac{(1-x)(1+x)}{x^2}\right| \\ &= \frac{1}{|x|^2} \cdot |x+1| \cdot |x-1| < 4(2.5)\delta < 10 \cdot \frac{\varepsilon}{10} = \varepsilon. \end{aligned}$$

   Therefore, f is continuous at $x = 1$. □

2. Use definition to prove that $f(x) = \dfrac{1}{x^2}$ is continuous at $x = -1$.

   Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{10}\}$ such that $|x+1| < \delta$. Then $|x+1| < 0.5$. So,

   $$-0.5 < x + 1 < 0.5.$$

   We obtain

   $$-1.5 < x < -0.5 \text{ and } -2.5 < x - 1 < -1.5.$$

   Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x-1| < 2.5$. We obtain

   $$\begin{aligned} |f(x) - f(-1)| &= \left|\frac{1}{x^2} - 1\right| = \left|\frac{1-x^2}{x^2}\right| = \left|\frac{(1-x)(1+x)}{x^2}\right| \\ &= \frac{1}{|x|^2} \cdot |x+1| \cdot |x-1| < 4(\delta)2.5 < 10 \cdot \frac{\varepsilon}{10} = \varepsilon. \end{aligned}$$

   Therefore, f is continuous at $x = -1$. □

3. Use definition to prove that $f(x) = \dfrac{2}{x^2}$ is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{20}\}$ such that $|x-1| < \delta$. Then $|x-1| < 0.5$. So,

$$-0.5 < x - 1 < 0.5.$$

We obtain

$$0.5 < x < 1.5 \text{ and } 1.5 < x + 1 < 2.5.$$

Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x+1| < 2.5$. We obtain

$$\begin{aligned} |f(x) - f(1)| &= \left|\frac{2}{x^2} - 2\right| = \left|\frac{2 - 2x^2}{x^2}\right| = \left|\frac{2(1-x)(1+x)}{x^2}\right| \\ &= \frac{1}{|x|^2} \cdot 2|x+1| \cdot |x-1| < 4(2.5)2\delta < 20 \cdot \frac{\varepsilon}{20} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 1$. □

4. Use definition to prove that $f(x) = \dfrac{2}{x^2}$ is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{20}\}$ such that $|x+1| < \delta$. Then $|x+1| < 0.5$. So,

$$-0.5 < x + 1 < 0.5.$$

We obtain

$$-1.5 < x < -0.5 \text{ and } -2.5 < x - 1 < -1.5.$$

Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x-1| < 2.5$. We obtain

$$\begin{aligned} |f(x) - f(-1)| &= \left|\frac{2}{x^2} - 2\right| = \left|\frac{2 - 2x^2}{x^2}\right| = \left|\frac{2(1-x)(1+x)}{x^2}\right| \\ &= \frac{1}{|x|^2} \cdot 2|x+1| \cdot |x-1| < 4(2\delta)2.5 < 20 \cdot \frac{\varepsilon}{20} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = -1$. □

5. Use definition to prove that $f(x) = \dfrac{1}{2x^2}$ is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{5}\}$ such that $|x - 1| < \delta$. Then $|x - 1| < 0.5$. So,

$$-0.5 < x - 1 < 0.5.$$

We obtain

$$0.5 < x < 1.5 \text{ and } 1.5 < x + 1 < 2.5.$$

Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x + 1| < 2.5$. We obtain

$$\begin{aligned}|f(x) - f(1)| &= \left|\frac{1}{2x^2} - \frac{1}{2}\right| = \left|\frac{1 - x^2}{2x^2}\right| = \left|\frac{(1 - x)(1 + x)}{2x^2}\right| \\ &= \frac{1}{2|x|^2} \cdot |x + 1| \cdot |x - 1| < 2(2.5)\delta < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.\end{aligned}$$

Therefore, f is continuous at $x = 1$. □

6. Use definition to prove that $f(x) = \dfrac{1}{2x^2}$ is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{5}\}$ such that $|x + 1| < \delta$. Then $|x + 1| < 0.5$. So,

$$-0.5 < x + 1 < 0.5.$$

We obtain

$$-1.5 < x < -0.5 \text{ and } -2.5 < x - 1 < -1.5.$$

Thus, $\dfrac{1}{|x|^2} < 2^2$ and $|x - 1| < 2.5$. We obtain

$$\begin{aligned}|f(x) - f(-1)| &= \left|\frac{1}{2x^2} - \frac{1}{2}\right| = \left|\frac{1 - x^2}{2x^2}\right| = \left|\frac{(1 - x)(1 + x)}{2x^2}\right| \\ &= \frac{1}{2|x|^2} \cdot |x + 1| \cdot |x - 1| < 2(\delta)2.5 < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.\end{aligned}$$

Therefore, f is continuous at $x = -1$. □

No.2

1. Let $f(x) = x^2 - 1$ where $x \in [1,2]$ and $P = \left\{1 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1,2]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(1 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 1 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(1 + \frac{j}{n}\right) - \left(1 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{j}{n}\right)^2 - 1\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{2j}{n} + \frac{j^2}{n^2}\right) - 1\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{2j}{n} + \frac{j^2}{n^2}\right] = \frac{1}{n^2}\left[2\sum_{j=1}^{n} j + \frac{1}{n}\sum_{j=1}^{n} j^2\right] \\
&= \frac{1}{n^2}\left[n(n+1) + \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} = 1 + \frac{1}{3} = \frac{4}{3} \quad \#$$

2. Let $f(x) = x^2 + 1$ where $x \in [1,2]$ and $P = \left\{1 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1,2]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(1 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 1 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(1 + \frac{j}{n}\right) - \left(1 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{j}{n}\right)^2 + 1\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{2j}{n} + \frac{j^2}{n^2}\right) + 1\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{2j}{n} + \frac{j^2}{n^2} + 2\right] = \frac{1}{n}\left[\frac{2}{n}\sum_{j=1}^{n} j + \frac{1}{n^2}\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 2\right] \\
&= \frac{1}{n}\left[\frac{2}{n}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + 2n\right] \\
&= \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} + 2
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} + 2 = 1 + \frac{1}{3} + 2 = \frac{10}{3} \quad \#$$

3. Let $f(x) = x^2 - 2$ where $x \in [1, 2]$ and $P = \left\{1 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[1, 2]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(1 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 1 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(1 + \frac{j}{n}\right) - \left(1 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{j}{n}\right)^2 - 2\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(1 + \frac{2j}{n} + \frac{j^2}{n^2}\right) - 2\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{2j}{n} + \frac{j^2}{n^2} - 1\right] = \frac{1}{n}\left[\frac{2}{n}\sum_{j=1}^{n} j + \frac{1}{n^2}\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} 1\right] \\
&= \frac{1}{n}\left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + \frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - n\right] \\
&= \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} - 1
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} - 1 = 1 + \frac{1}{3} - 1 = \frac{1}{3} \quad \#$$

4. Let $f(x) = x^2 - 1$ where $x \in [2, 3]$ and $P = \left\{2 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2, 3]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(2 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 2 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(2 + \frac{j}{n}\right) - \left(2 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(2 + \frac{j}{n}\right)^2 - 1\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(4 + \frac{4j}{n} + \frac{j^2}{n^2}\right) - 1\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{4j}{n} + \frac{j^2}{n^2} + 3\right] = \frac{1}{n}\left[\frac{4}{n}\sum_{j=1}^{n} j + \frac{1}{n^2}\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 3\right] \\
&= \frac{1}{n}\left[\frac{4}{n} \cdot \frac{n(n+1)}{2} + \frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + 3n\right] \\
&= \frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 3
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 3 = 2 + \frac{1}{3} + 3 = \frac{16}{3} \quad \#$$

5. Let $f(x) = x^2 + 1$ where $x \in [2,3]$ and $P = \left\{2 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2,3]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(2 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 2 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(2 + \frac{j}{n}\right) - \left(2 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(2 + \frac{j}{n}\right)^2 + 1\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(4 + \frac{4j}{n} + \frac{j^2}{n^2}\right) + 1\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{4j}{n} + \frac{j^2}{n^2} + 5\right] = \frac{1}{n}\left[\frac{4}{n}\sum_{j=1}^{n} j + \frac{1}{n^2}\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 5\right] \\
&= \frac{1}{n}\left[\frac{4}{n}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + 5n\right] \\
&= \frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 5
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 5 = 2 + \frac{1}{3} + 5 = \frac{22}{3} \quad \#$$

6. Let $f(x) = x^2 - 2$ where $x \in [2,3]$ and $P = \left\{2 + \frac{j}{n} : j = 0, 1, ..., n\right\}$ be a partition of $[2,3]$. Find $I(f)$.

Solution. Choose $f(t_j) = f(2 + \frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ where $x_j = 2 + \frac{j}{n}$ for all $j = 0, 1, ..., n$. Then

$$\Delta x_j = x_j - x_{j-1} = \left(2 + \frac{j}{n}\right) - \left(2 + \frac{j-1}{n}\right) = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(1 + \frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(2 + \frac{j}{n}\right)^2 - 2\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(4 + \frac{4j}{n} + \frac{j^2}{n^2}\right) - 2\right] = \frac{1}{n}\sum_{j=1}^{n}\left[\frac{4j}{n} + \frac{j^2}{n^2} + 2\right] = \frac{1}{n}\left[\frac{4}{n}\sum_{j=1}^{n} j + \frac{1}{n^2}\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 2\right] \\
&= \frac{1}{n}\left[\frac{4}{n}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + 2n\right] \\
&= \frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 2
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{2(n+1)}{n} + \frac{(n+1)(2n+1)}{6n^2} + 2 = 2 + \frac{1}{3} + 2 = \frac{13}{3} \quad \#$$