QUIZ 1 : MAP2406 Mathematical Analysis

Topic Field axioms, Well-Ordering Principle and Completeness axioms **Score** 10 points
Time Friday 25 Jan 2019, 3rd Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Let $x, y \in \mathbb{R}$. Show that

$$\text{if} \quad 0 < x < y, \quad \text{then} \quad 0 < x^3 < y^3.$$

2. **(3 points)** Prove that $|x-1| \leq 2$ implies $|x^2 - 5x + 6| \leq 4|x-2|$

3. **(4 points)** Let $A = \left\{ \dfrac{3n}{n+1} : n \in \mathbb{N} \right\}$. Find **sup** $\boldsymbol{A}$ and prove it.

Solution QUIZ 1 : MAP2406 Mathematical Analysis

Topic	Field axioms, Well-Ordering Principle and Completeness axioms	**Score**	10 points
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1. **(3 points)** Let $x, y \in \mathbb{R}$. Show that

$$\text{if} \quad 0 < x < y, \quad \text{then} \quad 0 < x^3 < y^3.$$

Proof. Let $x, y \in \mathbb{R}$. Assume that $0 < x < y$. Then

$$0 < x^3 < x^2y \quad \text{and} \quad 0 < xy^2 < y^3.$$

Since xy is positive, $0 < x(xy) < y(xy)$. That is $x^2y < xy^2$. By transitive law,

$$0 < x^3 < y^3.$$

□

2. **(3 points)** Prove that $|x - 1| \leq 2$ implies $|x^2 - 5x + 6| \leq 4|x - 2|$

Proof. Assume that $|x - 1| \leq 2$. Then

$$\begin{aligned}
|x^2 - 5x + 6| &= |(x-2)(x-3)| \\
&= |x-2||x-3| \\
&= |x-2||(x-1)-2| \\
&\leq |x-2|(|x-1|+2) && \text{Triangle iequaity} \\
&\leq |x-2|(2+2) && \text{Assumption} \\
&= 4|x-2|
\end{aligned}$$

□

3. **(4 points)** Let $A = \left\{ \frac{3n}{n+1} : n \in \mathbb{N} \right\}$. Find **sup *A*** and prove it.

Solution. Consider

$$A = \left\{ \frac{3}{2}, \frac{6}{2}, \frac{9}{4}, \frac{15}{6}, \dots \right\}$$

Claim that $\sup A = 3$.

Proof. Since $n < n + 1$ for all $n \in \mathbb{N}$, $\frac{n}{n+1} < 1$. Then

$$\frac{3n}{n+1} < 3$$

Thus, 3 is an upper bound of A.
Let u_0 be upper bound of A such that $u_0 < 3$. Then $\frac{3 - u_0}{3} > 0$. By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{3 - u_0}{3}$$
$$u_0 < 3 - \frac{3}{n_0} = \frac{3(n_0 - 1)}{n_0}$$

If $n_0 = 1$, then $u_0 < 0$ imposible. So, $n_0 > 1$. Choose $n = n_0 - 1 > 1$. It implies that

$$u_0 < \frac{3n}{n+1}$$

Thus, u_0 is not upper bound of A. It is contradiction. □

QUIZ 2 : MAP2406 Mathematical Analysis

Topic Sequence in $\mathbb{R}$ **Score** 10 points
Time Tuesday 12 Feb 2019, 5th Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 marks)** Prove that $\dfrac{n^2}{n^2+1} \to 1$ as $n \to \infty$ by Definition.
2. **(4 marks)** Suppose that $x_n \to 1$ as $n \to \infty$. Use Definition to prove that
$$x_n^2 \to 1 \text{ as } n \to \infty.$$
3. **(3 marks)** Let $K > 0$ and $x_n \to \infty$ as $n \to \infty$. Prove that
$$Kx_n \to \infty \quad \text{as } n \to \infty$$

Solution QUIZ 2 : MAP2406 Mathematical Analysis

Topic Sequence in $\mathbb{R}$ **Score** 10 points
Time Tuesday 12 Feb 2019, 5th Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics,
Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 marks)** Prove that $\dfrac{n^2}{n^2+1} \to 1$ as $n \to \infty$ by Definition.

Proof. Let $\varepsilon > 0$. By Archimedean Principle, there $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \sqrt{\varepsilon}$.
For each $n \geq N$, i.e., $\dfrac{1}{n^2} \leq \dfrac{1}{N^2} < \varepsilon$, we obtain

$$\begin{aligned}\left|\frac{n^2}{n^2+1} - 1\right| &= \left|\frac{-1}{n^2+1}\right| \\ &= \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.\end{aligned}$$

□

2. **(4 marks)** Suppose that $x_n \to 1$ as $n \to \infty$. Use Definition to prove that

$$x_n^2 \to 1 \text{ as } n \to \infty.$$

Proof. Suppose that $x_n \to 1$ as $n \to \infty$. Since $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, i.e., there is $M > 0$ such that

$$|x_n| \leq M.$$

Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$|x_n - 1| < \frac{\varepsilon}{M+1}.$$

Since $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, i.e., there is $M \in \mathbb{R}$ such that

$$|x_n| \leq M.$$

For each $n \geq N$,

$$\begin{aligned}|x^2 + 1 - 1| &= |(x_n - 1)(x_n + 1)| \\ &= |x_n - 1||x_n + 1| \\ &< \frac{\varepsilon}{M+1} \cdot (|x_n| + 1) \\ &< \frac{\varepsilon}{M+1} \cdot (M+1) = \varepsilon\end{aligned}$$

□

3. **(3 marks)** Let $K > 0$ and $x_n \to \infty$ as $n \to \infty$. Prove that

$$Kx_n \to \infty \quad \text{as } n \to \infty$$

Proof. Let $K > 0$ and $x_n \to \infty$ as $n \to \infty$.
Let $M \in \mathbb{R}$. Set $M_1 = \dfrac{M}{K}$. There is an $N \in \mathbb{N}$, $n \geq N$ implies $x_n > M_1$.
For each $n \geq N$, it implies

$$Kx_n > KM_1 = M.$$

□

QUIZ 3 : MAP2406 Mathematical Analysis

Topic Diferentiation **Score** 10 points
Time Friday 22 Mar 2019, 12th Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics,
Faculty of Education, Suan Sunandha Rajabhat University

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1. **(4 marks)** If $\lim_{x\to 0} \dfrac{x}{\sin x} = 1$, use L'Hospital's rule to find

$$\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}}.$$

2. **(3 marks)** Use the Mean Value Theorem to prove that

$$\sqrt{x+1} \leq x+1 \quad \text{for all } x \geq 0.$$

3. **(3 marks)** Use the Inverse Function Theorem to find $(f^{-1})'(e)$ when

$$f(x) = x^2 e^{x^2}.$$

Solution QUIZ 3 : MAP2406 Mathematical Analysis

Topic Diferentiation **Score** 10 points
Time Friday 22 Mar 2019, 12th Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics,
Faculty of Education, Suan Sunandha Rajabhat University

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1. **(4 marks)** If $\lim_{x\to 0}\frac{x}{\sin x}=1$, use L'Hospital's rule to find

$$\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}}.$$

Solution. Set $L=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}}$. Then

$$\begin{aligned}
\ln L &= \ln\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}} \\
&= \lim_{x\to 0}\frac{\ln\left(\frac{x}{\sin x}\right)}{x^2} && I.F.\frac{0}{0} \\
&= \lim_{x\to 0}\frac{\frac{\sin x}{x}\cdot\frac{\sin x - x\cos x}{\sin^2 x}}{2x} && \text{L'Hospital's rule} \\
&= \lim_{x\to 0}\frac{\sin x - x\cos x}{2x^2\sin x} && I.F.\frac{0}{0} \\
&= \lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{4x\sin x+2x^2\cos x} && \text{L'Hospital's rule} \\
&= \lim_{x\to 0}\frac{\sin x}{4\sin x+2x\cos x} && I.F.\frac{0}{0} \\
&= \lim_{x\to 0}\frac{\cos x}{4\cos x+2\cos x-2x\sin x} && \text{L'Hospital's rule} \\
&= \frac{1}{6} \\
L &= e^{\frac{1}{6}} \quad \#
\end{aligned}$$

2. **(3 marks)** Use the Mean Value Theorem to prove that

$$\sqrt{x+1}\le x+1 \quad \text{for all } x\ge 0.$$

Proof. Let $f(x)=\sqrt{x+1}-x-1$ for $x\ge 0$. Then $f'(x)=\frac{1}{2\sqrt{x+1}}-1$.
By the Mean Value Theorem, there is $c\in[0,x]$ such that

$$\begin{aligned}
f(x)-f(0) &= f'(c)(x-0) \\
\sqrt{x+1}-x-1 &= \left(\frac{1}{2\sqrt{c+1}}-1\right)\cdot x
\end{aligned}$$

Since $c>0$, $c+1>1$. So, $2\sqrt{c+1}>2$. Then

$$\begin{aligned}
\frac{1}{2\sqrt{c+1}} &< \frac{1}{2} \\
\frac{1}{2\sqrt{c+1}}-1 &< \frac{1}{2}-1<0
\end{aligned}$$

Therefore,

$$\sqrt{x+1}-x-1\le 0 \quad \text{for all } x\ge 0.$$

□

3. **(3 marks)** Use the Inverse Function Theorem to find $(f^{-1})'(e)$ when

$$f(x) = x^2 e^{x^2}.$$

Solution. Then $f'(x) = 4x^3 e^{x^2} + 2xe^{x^2}$. Since $f(1) = e$, $f^{-1}(e) = 1$. By the Inverse Function Theorem,

$$\begin{aligned}
(f^{-1})'(e) &= \frac{1}{f'(f^{-1}(e))} \\
&= \frac{1}{f'(1)} \\
&= \frac{1}{4e + 2e} \\
&= \frac{1}{6e} \quad \#
\end{aligned}$$

QUIZ 4 : MAP2406 Mathematical Analysis

Topic Integrability **Score** 10 points
Time Friday 29 Mar 2019, 13th Week, Semester 2/2018
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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1. **(4 marks)** Let $f(x) = 1 - ax^2$ on $[-1, 1]$ where $0 < a < 1$.

$$\text{If } L(f,P) = \frac{11}{8}, \quad \text{find } a \text{ and } U(f,P) \text{ when } P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

2. **(3 marks)** If $f(x) = \displaystyle\int_0^{x^2} \cos(t^2)\,dt$, show that

$$3\int_0^1 \cos(x^2)\,dx - 6\int_0^1 xf(x)\,dx = 2\sin 1.$$

3. **(3 marks)** Define

$$f(x) = \begin{cases} x-1 & \text{if} \quad 0 \le x \le 1 \\ x-2 & \text{if} \quad 1 < x \le 2 \\ x-3 & \text{if} \quad 2 < x \le 3 \\ & \vdots \\ x-n & \text{if} \quad n-1 < x \le n \end{cases}$$

Suppose that $\displaystyle\int_0^n f(x)\,dx = 2019$. What is a number n ?

Solution QUIZ 4 : MAP2406 Mathematical Analysis

Topic	Integrability **Score** 10 points
Time	Friday 29 Mar 2019, 13th Week, Semester 2/2018
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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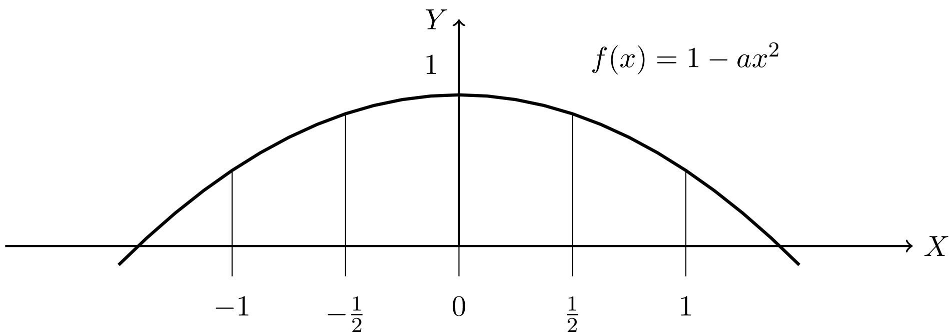
1. **(4 marks)** Let $f(x) = 1 - ax^2$ on $[-1, 1]$ where $0 < a < 1$.

$$\text{If } L(f,P) = \frac{11}{8}, \text{ find } a \text{ and } U(f,P) \text{ when } P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

Solution.



$$\begin{aligned}
L(f,P) &= \frac{1}{2}f(-1) + \frac{1}{2}f\left(-\frac{1}{2}\right) + \frac{1}{2}f\left(-\frac{1}{2}\right) + \frac{1}{2}f(1) \\
\frac{11}{8} &= \frac{1}{2}\left[(1-a) + \left(1 - \frac{1}{4}a\right) + \left(1 - \frac{1}{4}a\right) + (1-a)\right] \\
\frac{11}{4} &= 4 - \frac{5}{2}a \\
11 &= 16 - 10a \\
\therefore \quad a &= \frac{1}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
U(f,P) &= \frac{1}{2}f\left(-\frac{1}{2}\right) + \frac{1}{2}f(0) + \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\left[\frac{7}{8} + 1 + 1 + \frac{7}{8}\right] \\
&= \frac{15}{8} \quad \#
\end{aligned}$$

2. **(3 marks)** If $f(x) = \int_0^{x^2} \cos(t^2)\, dt$, show that

$$2\int_0^1 \cos(x^2)\, dx - 4\int_0^1 xf(x)\, dx = \sin 1.$$

Solution. Then $f'(x) = 2x\cos(x^4)$ and $f(1) = \int_0^1 \cos(x^2)\, dx$. Apply integration by part,

$$\begin{aligned}
\int_0^1 x^2 f'(x)\, dx &= [x^2 f(x)]_0^1 - \int_0^1 (2x)f(x)\, dx \\
\int_0^1 x^2 2x\cos(x^4)\, dx &= f(1) - \int_0^1 (2x)f(x)\, dx \\
\frac{1}{2}[\sin(x^4)]_0^1 &= \int_0^1 \cos(x^2)\, dx - 2\int_0^1 xf(x)\, dx \\
\frac{1}{2}\sin 1 &= \int_0^1 \cos(x^2)\, dx - 2\int_0^1 xf(x)\, dx \\
\sin 1 &= 2\int_0^1 \cos(x^2)\, dx - 4\int_0^1 xf(x)\, dx
\end{aligned}$$

3. **(3 marks)** Define

$$f(x) = \begin{cases} x-1 & \text{if} \quad 0 \le x \le 1 \\ x-2 & \text{if} \quad 1 < x \le 2 \\ x-3 & \text{if} \quad 2 < x \le 3 \\ & \vdots \\ x-n & \text{if} \quad n-1 < x \le n \end{cases}$$

Suppose that $\int_0^n f(x)\, dx = -2019$. What is a number n ?

Solution.

$$\begin{aligned}
-2019 &= \int_0^n f(x)\, dx \\
&= \int_0^1 (x-1)dx + \int_1^2 (x-2)dx + \int_2^3 (x-3)dx + \cdots + \int_{n-1}^n (x-n)dx \\
&= \left[\frac{(x-1)^2}{2}\right]_0^1 + \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-3)^2}{2}\right]_2^3 + \cdots + \left[\frac{(x-n)^2}{2}\right]_{n-1}^n \\
&= -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \cdots - \frac{1}{2} \\
&= -\frac{1}{2}n
\end{aligned}$$

$$\therefore \quad n = 2019 \cdot 2 = 4038 \quad \#$$