QUIZ 1 (Sec1-2): MAP2406 Mathematical Analysis

Topic Field axioms and Completeness axioms **Score** 10 points
Time Thurday 30 Jan 2020, 3rd Week, Semester 2/2019
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics,
Faculty of Education, Suan Sunandha Rajabhat University

NAME... **ID**.................................... **SECTION**.......................

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1. **(3 points)** Let $x, y \in \mathbb{R}$. Show that
$$x^2 + y^2 \geq xy$$

2. **(3 points)** Let $a, b, c \in \mathbb{R}$. Use Triangle inequality to prove that
$$|a - c| \leq |a - b| + |b - c|.$$

3. **(4 points)** Let $A = \left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$. Find **sup** $\boldsymbol{A}$ and prove it.

Solution QUIZ 1 : MAP2406 Mathematical Analysis

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1. **(3 points)** Let $x, y \in \mathbb{R}$. Show that
$$x^2 + y^2 \geq xy$$

Proof. Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned} x^2 + y^2 - xy &= x^2 - xy + y^2 \\ &= x^2 - 2x\left(\frac{y}{2}\right) + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2 + y^2 \\ &= \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4} \geq 0 \end{aligned}$$

Therefore, $x^2 + y^2 \geq xy$. □

2. **(3 points)** Let $a, b, c \in \mathbb{R}$. Use Triangle inequality to prove that

$$|a - c| \leq |a - b| + |b - c|.$$

Proof. Let $a, b, c \in \mathbb{R}$. By Triangle inequality ,

$$|a - c| = |(a - b) + (b - c)| \leq |a - c| + |b - c|$$

□

3. **(4 points)** Let $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Find **sup *A*** and prove it.

Solution. Consider
$$A = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...\right\}$$

Claim that $\sup A = 1$.

Proof. Let $n \in \mathbb{N}$. Then $n > 0$. So, $\frac{1}{n} > 0$ or $-\frac{1}{n} < 0$. Thus,

$$1 - \frac{1}{n} \leq 1$$

Thus, 1 is an upper bound of A.
Let u_0 be an upper bound of A such that $u_0 < 1$. Then $1 - u_0 > 0$.
By Achimedean principle, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < 1 - u_0$$
$$u_0 < 1 - \frac{1}{n_0}$$

So, u_0 is not an upper bound of A. It is contradiction. □

QUIZ 1 (Sec1-2): MAP2406 Mathematical Analysis

Topic Field axioms and Completeness axioms **Score** 10 points
Time Thurday 5 Feb 2020, 3rd Week, Semester 2/2019
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics,
Faculty of Education, Suan Sunandha Rajabhat University

NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Let $x, y \in \mathbb{R}$. Show that
$$x + y \geq \sqrt{xy}$$

2. **(3 points)** Let $a, b, c \in \mathbb{R}$. Use Triangle inequality to prove that
$$|a - b| - |b - c| \leq |a - c|.$$

3. **(4 points)** Let $A = \left\{ 1 + \dfrac{1}{n} : n \in \mathbb{N} \right\}$. Find **inf** $\boldsymbol{A}$ and prove it.

QUIZ 2 (Sec1-2): MAP2406 Mathematical Analysis

Topic Limit of Sequences and Cauchy Sequences **Score** 10 points
Time Thursday 13 Feb 2020, 5rd Week, Semester 2/2019
Teacher Assistant Professor Thanatyod Jampawai, Ph.D., Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Use definition to prove that $\lim\limits_{n\to\infty} \dfrac{2n^2-1}{n^2+1} = 2$
2. **(4 points)** Assume that $x_n \to -1$ as $n \to \infty$. Show that $(x_n - 1)^2 \to 4$ as $n \to \infty$.
3. **(3 points)** Prove that $\left\{\dfrac{n^2-1}{n^2+1}\right\}$ is a Cauchy sequence.

Solution QUIZ 2: MAP2406 Mathematical Analysis

Topic Limit of Sequences and Cauchy Sequences **Score** 10 points
Time Thursday 13 Feb 2020, 5rd Week, Semester 2/2019
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1. **(3 points)** Use definition to prove that $\lim_{n\to\infty} \frac{2n^2-1}{n^2+1} = 2$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\frac{\varepsilon}{3}} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\frac{\varepsilon}{3}}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. So, $n^2 \geq N^2$. Then $\frac{1}{n^2} \leq \frac{1}{N^2}$. We see that $n^2+1 > n^2$.
It follows that $\frac{1}{n^2+1} \leq \frac{1}{n^2}$. We obtain

$$\begin{aligned}\left|\frac{2n^2-1}{n^2+1} - 2\right| &= \left|\frac{2n^2-1-2(n^2+1)}{n^2+1}\right| = \frac{3}{n^2+1} \\ &< \frac{3}{n^2} \leq \frac{3}{N^2} < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{2n^2-1}{n^2+1} = 2.$ □

2. **(4 points)** Assume that $x_n \to -1$ as $n \to \infty$. Show that $(x_n - 1)^2 \to 4$ as $n \to \infty$.

Proof. Assume that $x_n \to -1$ as $n \to \infty$.
Given $\varepsilon = 1$. There is an $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implying $|x_n + 1| < 1$. Then

$$\begin{aligned}|x_n| - 1 \leq |x_n + 1| &\leq 1 \\ |x_n| &\leq 2\end{aligned}$$

Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|x_n + 1| < \frac{\varepsilon}{5}$.
Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$. We obtain

$$\left|(x_n-1)^2 - 4\right| = |(x_n+1)(x_n-3)| \leq |x_n+1|(|x_n|+3) < \frac{\varepsilon}{5}\cdot(2+3) = \varepsilon$$

Thus, $(x_n)^2 \to 4$ as $n \to \infty$ □

3. **(3 points)** Prove that $\left\{\frac{n^2-1}{n^2+1}\right\}$ is a Cauchy sequence.

Proof. Let $\varepsilon > 0$. Then $\frac{\sqrt{\varepsilon}}{2} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\sqrt{\varepsilon}}{2}$.
Let $n \in \mathbb{N}$ such that $n \geq N$ and $m \geq N$. Then $n^2 \geq N^2$ and $m^2 \geq N^2$. So, $\frac{1}{n^2} \leq \frac{1}{N^2}$ and $\frac{1}{m^2} \leq \frac{1}{N^2}$.
We obtain

$$\begin{aligned}\left|\frac{n^2-1}{n^2+1} - \frac{m^2-1}{m^2+1}\right| &= \left|\frac{(n^2-1)(m^2+1)-(m^2-1)(n^2+1)}{(n^2+1)(m^2+1)}\right| = \left|\frac{2n^2-2m^2}{(n^2+1)(m^2+1)}\right| \\ &= \left|\frac{2n^2}{(n^2+1)(m^2+1)} - \frac{2m^2}{(n^2+1)(m^2+1)}\right| \\ &= \frac{2n^2}{(n^2+1)(m^2+1)} + \frac{2m^2}{(n^2+1)(m^2+1)} \\ &\leq \frac{2n^2}{(n^2)(m^2)} + \frac{2m^2}{(n^2)(m^2)} = \frac{2}{m^2} + \frac{2}{n^2} \leq \frac{2}{N^2} + \frac{2}{N^2} = \frac{4}{N^2} < \varepsilon.\end{aligned}$$

Thus, $\left\{\frac{n^2}{n^2+1}\right\}$ is Cauchy. □