QUIZ 1 : MAT2303 ABSTRACT ALGEBRA

TOPIC Groups & Symmetric groups **SCORE** 10 points
QUIZ TIME Thu 31 Aug 2017, 3rd Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Define $a * b = 7ab$ for all $a, b \in \mathbb{Q}^+$. Prove that $(\mathbb{Q}^+, *)$ is a group.

2. **(3 points)** Compute **inverses** for each element in the following groups.

   2.1 543 in $(\mathbb{Z}_{2560}, +)$ 2.2 7 in $(\mathbb{Z}_{15}^{\times}, \cdot)$ 2.3 $(1\,2\,3\,4)^3$ in $(S_4, \circ)$

3. **(4 points)** Compute **orders** for each element in the following groups.

   3.1 9 in $(\mathbb{Z}_{144}, +)$ 3.3 $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ in $(\mathbb{C} - \{0\}, \cdot)$

   3.2 5 in $(\mathbb{Z}_{21}^{\times}, \cdot)$ 3.4 $(1\,2\,4\,3)(3\,4\,2\,5)$ in $(S_5, \circ)$

SOLUTION QUIZ 1 : MAT2303 (SEC2)

TOPIC	Groups & Symmetric groups	**SCORE**	10 points
QUIZ TIME	Thu 31 Aug 2017, 3rd Week, Semester 1/2016		
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University		

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1. **(3 points)** Define $a * b = 7ab$ for all $a, b \in \mathbb{Q}^+$. Prove that $(\mathbb{Q}^+, *)$ is a group.

   Proof. First, let $a, b, c \in \mathbb{Q}^+$. Then

   $$(a * b) * c = (7ab) * c = 7(7ab)c = 7(a)(7bc) = a * (7bc) = a * (b * c).$$

   So, $*$ is associative on $\mathbb{Q}^+$. Next, let $a \in \mathbb{Q}^+$. We obtain

   $$a * \frac{1}{7} = 7a(\frac{1}{7}) = a = 7(\frac{1}{7})a = \frac{1}{7} * a.$$

   Thus, $\frac{1}{7}$ is the identity. Finally, we will prove that all elements in $\mathbb{Q}^+$ have inverses. Let $a \in \mathbb{Q}^+$. Since a in a nonzero rational number, $\frac{1}{49a}$ is a positive rational number. We get

   $$a * (\frac{1}{49a}) = 7a(\frac{1}{49a}) = \frac{1}{7} = 7(\frac{1}{49a})a = \frac{1}{49a} * a.$$

   Hence, $\frac{1}{49a}$ ia an inverses of a. Therefore, $(\mathbb{Q}^+, *)$ is a group. □

2. **(3 points)** Compute **inverses** for each element in the following groups.

   2.1 543 in $(\mathbb{Z}_{2560}, +)$
   Since $543 + 2017 = 2560 = 0$, 2017 is the inverse of 345.

   2.2 7 in $(\mathbb{Z}_{15}^{\times}, \cdot)$
   Since $7 \cdot 13 = 91 = 1$, 13 is the inverse of 7.

   2.3 $(1\,2\,3\,4)^3$ in $(S_4, \circ)$
   Let $\alpha = (1\,2\,3\,4)^3$. Then,

   $$\alpha = (1\,2\,3\,4)^3 = (1\,2\,3\,4)(1\,2\,3\,4)(1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$
   $$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\,2\,3\,4)$$

   Hence, $(1\,2\,3\,4)$ is the inverse of $(1\,2\,3\,4)^3$.

3. **(4 points)** Compute **orders** for each element in the following groups.

   3.1 9 in $(\mathbb{Z}_{144}, +)$
   Since $16(9) = 144 = 0$, $\circ(9) = 16$.

   3.2 5 in $(\mathbb{Z}_{21}^{\times}, \cdot)$
   Consider,

   $$\begin{aligned} 5^2 &= 25 = 4 \\ 5^3 &= 5(4) = 20 \\ 5^4 &= 5(20) = 100 = 16 \\ 5^5 &= 5(16) = 80 = 17 \\ 5^6 &= 5(17) = 85 = 1 \end{aligned}$$

   Thus, $\circ(5) = 6$

3.3 $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ in $(\mathbb{C}-\{0\},\cdot)$
Consider,

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2=\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right)+\left(i\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+i\frac{\sqrt{3}}{2}-\frac{3}{4}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$=\left(i\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^2=-1$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(-1)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2=(-1)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^6=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3=(-1)(-1)=1$$

Hence, $\circ(\frac{1}{2}+i\frac{\sqrt{3}}{2})=6$

3.4 $(1\,2\,4\,3)(3\,4\,2\,5)$ in $(S_5,\circ)$
Let $\alpha=(1\,2\,4\,3)(3\,4\,2\,5)$. Then

$$\alpha=(1\,2\,4\,3)(3\,4\,2\,5)=\begin{pmatrix}1&2&3&4&5\\5&2&1&4&3\end{pmatrix}$$
$$\alpha^2=\begin{pmatrix}1&2&3&4&5\\5&2&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\5&2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\3&2&5&4&1\end{pmatrix}$$
$$\alpha^3=\alpha\alpha^2=\begin{pmatrix}1&2&3&4&5\\5&2&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\3&2&5&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\1&2&3&4&5\end{pmatrix}=(1)$$

Thus, $\circ(\alpha)=3$.

QUIZ 1 : MAT2303 ABSTRACT ALGEBRA

TOPIC Groups & Symmetric groups **SCORE** 10 points
QUIZ TIME Fri 1 Sep 2017, 3rd Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME... **ID**.................................... **SECTION**.......................

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1. **(3 points)** Define $a * b = a + b - 387$ for all $a, b \in \mathbb{Z}$. Prove that $(\mathbb{Z}, *)$ is a group.

2. **(3 points)** Compute **inverses** for each element in the following groups.

   2.1 1112 in $(\mathbb{Z}_{2017}, +)$ 2.2 14 in $(\mathbb{Z}_{15}^{\times}, \cdot)$ 2.3 $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ in $(GL_2(\mathbb{R}), \cdot)$

3. **(4 points)** Compute **orders** for each element in the following groups.

   3.1 21 in $(\mathbb{Z}_{144}, +)$

   3.2 11 in $(\mathbb{Z}_{25}^{\times}, \cdot)$

   3.3 $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ in $(\mathbb{C} - \{0\}, \cdot)$

   3.4 $(1\,2\,5\,3)(4\,3\,2\,5)$ in $(S_5, \circ)$

SOLUTION QUIZ 1 : MAT2303 (SEC1)

TOPIC Groups & Symmetric groups **SCORE** 10 points
QUIZ TIME Fri 1 Sep 2017, 3rd Week, Semester 1/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University

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1. **(3 points)** Define $a * b = a + b - 387$ for all $a, b \in \mathbb{Z}$. Prove that $(\mathbb{Z}, *)$ is a group.

Proof. First, let $a, b, c \in \mathbb{Z}$. Then

$$\begin{aligned}(a * b) * c &= (a + b - 387) * c \\ &= (a + b - 387) + c - 387 \\ &= a + (b + c - 387) - 387 \\ &= a * (b + c - 387) \\ &= a * (b * c).\end{aligned}$$

So, $*$ is associative on $\mathbb{Z}$. Next, let $a \in \mathbb{Z}$. We obtain

$$a * 387 = a + 387 - 387 = a = 387 + a - 387 = 387 * a.$$

Thus, 387 is the identity. Finally, we will prove that all elements in $\mathbb{Z}$ have inverses. Let $a \in \mathbb{Z}$. We obtain

$$a * (774 - a) = a + (774 - a) - 387 = 387 = a + (774 - a) - 387 = (774 - a) * a.$$

Hence, $774 - a$ ia an inverses of a. Therefore, $(\mathbb{Z}, *)$ is a group. □

2. **(3 points)** Compute **inverses** for each element in the following groups.

2.1 1112 in $(\mathbb{Z}_{2017}, +)$
Since $1112 + 905 = 2017 = 0$, 905 is the inverse of 1113.

2.2 14 in $(\mathbb{Z}_{15}^{\times}, \cdot)$
Since $14 \cdot 14 = 196 = 1$, 14 is the inverse of 14.

2.3 $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$ in $(GL_2(\mathbb{R}), \cdot)$
Since $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$ is the inverse of $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$.

3. **(4 points)** Compute **orders** for each element in the following groups.

3.1 21 in $(\mathbb{Z}_{144}, +)$
Since $48(21) = 144(7) = 0$, $\circ(21) = 48$.

3.2 11 in $(\mathbb{Z}_{25}^{\times}, \cdot)$
Consider,

$$\begin{aligned}11^2 &= 121 = 21 \\ 11^3 &= 11(21) = 231 = 6 \\ 11^4 &= 11(6) = 66 = 16 \\ 11^5 &= 11(16) = 176 = 1\end{aligned}$$

Thus, $\circ(11) = 5$

3.3 $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ in $(\mathbb{C} - \{0\}, \cdot)$
Consider,

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) + \left(i\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - i\frac{\sqrt{3}}{2} - \frac{3}{4} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\left[\left(\frac{1}{2}\right)^2 - \left(i\frac{\sqrt{3}}{2}\right)^2\right] = -1$$

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^4 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(-1) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^5 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = (-1)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^6 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = (-1)(-1) = 1$$

Hence, $\circ(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 6$

3.4 $(1\,2\,5\,3)(4\,3\,2\,5)$ in $(S_5, \circ)$ Let $\alpha = (1\,2\,5\,3)(4\,3\,2\,5)$. Then

$$\alpha = (1\,2\,5\,3)(4\,3\,2\,5) = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 5 & 1 & 4\end{pmatrix}$$

$$\alpha^3 = \alpha\alpha^2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 5 & 1 & 4\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & 2 & 5 & 3\end{pmatrix}$$

$$\alpha^4 = \alpha\alpha^3 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & 2 & 5 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 5 & 4 & 2 & 1\end{pmatrix}$$

$$\alpha^5 = \alpha\alpha^4 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 5 & 4 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 3 & 4 & 5\end{pmatrix} = (1)$$

Thus, $\circ(\alpha) = 5$.

QUIZ 2 : MAT2303 ABSTRACT ALGEBRA

TOPIC Subgroups & Lagrange's theorem **SCORE** 10 points
QUIZ TIME Thu 14 Sep 2017, 5th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University

NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \text{ and } ab > 0 \right\}$. Prove that H is a **subgroup** of $GL_2(\mathbb{R})$.

2. **(3 points)** Compute

   2.1 the **number of all generators** of $\mathbb{Z}_{43200}$

   2.2 **index** of $[\mathbb{Z}_{9000} : \langle 150 \rangle]$

3. **(4 points)** Find **all subgroups** and draw **lattice diagram** of the following groups

   3.1 $\mathbb{Z}_{36}$

   3.2 $\mathbb{Z}_{27}^{\times}$

SOLUTION QUIZ 2 : MAT2303 (SEC2)

TOPIC	Subgroups & Lagrange's theorem	**SCORE**	10 points
QUIZ TIME	Thu 14 Sep 2017, 5th Week, Semester 1/2017		
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University		

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1. **(3 points)** Let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \text{ and } ab > 0 \right\}$. Prove that H is a **subgroup** of $GL_2(\mathbb{R})$.

   Proof. We first choose $a = b = 1$, so $ab = 1 > 0$. Then, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ belongs to H.

   Next, we will show that H is closed. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ be elements in H. Then

   $$AB = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} ax & 0 \\ 0 & by \end{bmatrix}$$

   Since $A, B \in H$, $ab > 0$ and $xy > 0$. We conclude that $(ax)(by) = (ab)(xy) > 0$. Thus, $AB \in H$.

   Finally, let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be in H. Then $ab > 0$. It follows that $a \neq 0$ and $b \neq 0$. Choose $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$. Then

   $$AA^{-1} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = A^{-1}A.$$

   It is easy to see that $\frac{1}{ab} > 0$. Hence, A^{-1} is an inverse of A and belongs to H. Therefore, $H \leq GL_2(\mathbb{R})$. □

2. **(3 points)** Compute

   2.1 the **number of all generators** of $\mathbb{Z}_{43200}$

   $$\begin{aligned} \text{The number of all generators of } \mathbb{Z}_{43200} &= \phi(43200) \\ &= \phi(2^6 \cdot 3^3 \cdot 5^2) \\ &= \phi(2^6)\phi(3^3)\phi(5^2) \\ &= (2^6 - 2^5)(3^3 - 3^2)(5^2 - 5) \\ &= (32)(18)(20) \\ &= 11520 \quad \# \end{aligned}$$

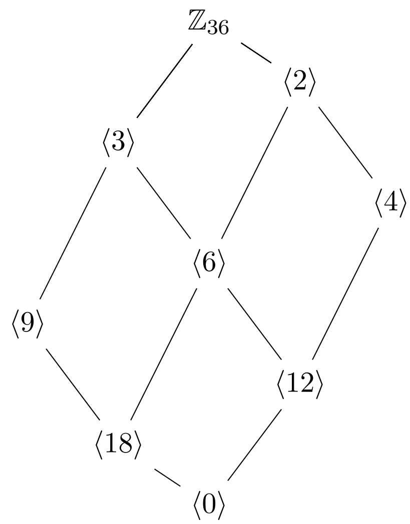
   2.2 **index** of $[\mathbb{Z}_{9000} : \langle 150 \rangle]$
   Consider $\langle 150 \rangle = \{0, 150, 300, 450, ..., 9000\}$. Then $\circ(\langle 150 \rangle) = 60$. Thus

   $$[\mathbb{Z}_{9000} : \langle 150 \rangle] = \frac{\circ(\mathbb{Z}_{9000})}{\circ(\langle 150 \rangle)} = \frac{9000}{60} = 150 \quad \#$$

3. **(4 points)** Find **all subgroups** and draw **lattice diagram** of the following groups

   3.1 $\mathbb{Z}_{36}$
   We choose 1 to be a generator of $\mathbb{Z}_{36}$. Then all divisors pf 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36. Hence, all subgroups are

   $\langle 1 \rangle = \mathbb{Z}_{36}$
   $\langle 2 \rangle = \{0, 2, 4, ..., 34\}$
   $\langle 3 \rangle = \{0, 3, 6, ..., 33\}$
   $\langle 4 \rangle = \{0, 4, 8, ..., 32\}$
   $\langle 6 \rangle = \{0, 6, 12, 18, 24, 30\}$
   $\langle 9 \rangle = \{0, 9, 18, 27\}$
   $\langle 12 \rangle = \{0, 12, 24\}$
   $\langle 18 \rangle = \{0, 18\}$
   $\langle 0 \rangle = \{0\}$



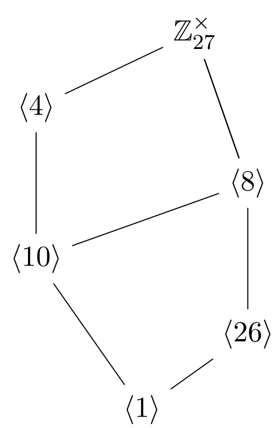
3.2 $\mathbb{Z}_{27}^{\times}$

For $\mathbb{Z}_{27}^{\times} = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$, we consider

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32 = 5$
$2^6 = 10$
$2^7 = 20$
$2^8 = 40 = 13$
$2^9 = 26$
$2^{10} = 52 = 25$
$2^{11} = 50 = 23$
$2^{12} = 46 = 19$
$2^{13} = 38 = 11$
$2^{14} = 22$
$2^{15} = 44 = 17$
$2^{16} = 34 = 7$
$2^{17} = 14$
$2^{18} = 28 = 1$

Hence, $\langle 2 \rangle = \mathbb{Z}_{27}^{\times}$. That is 2 to be a generator of $\mathbb{Z}_{27}^{\times}$. Since $\circ(\mathbb{Z}_{27}^{\times}) = 18$, all divisors of 18 are 1, 2, 3, 6, 9, 18. Then,

$$\left\langle 2^{\frac{18}{1}} \right\rangle = \left\langle 2^{18} \right\rangle = \langle 1 \rangle = \{1\}$$
$$\left\langle 2^{\frac{18}{2}} \right\rangle = \left\langle 2^{9} \right\rangle = \langle 26 \rangle = \{1, 26\}$$
$$\left\langle 2^{\frac{18}{3}} \right\rangle = \left\langle 2^{6} \right\rangle = \langle 10 \rangle = \{1, 10, 19\}$$
$$\left\langle 2^{\frac{18}{6}} \right\rangle = \left\langle 2^{3} \right\rangle = \langle 8 \rangle = \{1, 8, 10, 26, 19, 17\}$$
$$\left\langle 2^{\frac{18}{9}} \right\rangle = \left\langle 2^{2} \right\rangle = \langle 4 \rangle = \{1, 4, 16, 10, 13, 25, 19, 22, 7\}$$
$$\left\langle 2^{\frac{18}{18}} \right\rangle = \left\langle 2^{1} \right\rangle = \langle 2 \rangle = \mathbb{Z}_{27}^{\times}$$



QUIZ 2 : MAT2303 ABSTRACT ALGEBRA

TOPIC Subgroups & Lagrange's theorem **SCORE** 10 points
QUIZ TIME Fri 15 Sep 2017, 5th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME.. **ID**...................................... **SECTION**.........................

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1. **(3 points)** Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} : a, b \in \mathbb{R} \text{ and } a \neq 0 \right\}$. Prove that H is a **subgroup** of $GL_2(\mathbb{R})$.

2. **(3 points)** Compute

   2.1 the **number of all generators** of $\mathbb{Z}_{86400}$

   2.2 **index** of $[S_6 : \langle (1\,2\,3\,4)(4\,3\,5\,1) \rangle]$

3. **(4 points)** Find **all subgroups** and draw **lattice diagram** of the following groups

   3.1 $\mathbb{Z}_{24}$

   3.2 $\mathbb{Z}_{27}^{\times}$

SOLUTION QUIZ 2 : MAT2303 (SEC1)

TOPIC	Subgroups & Lagrange's theorem	**SCORE**	10 points
QUIZ TIME	Fri 15 Sep 2017, 5th Week, Semester 1/2017		
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University		

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1. **(3 points)** Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} : a, b \in \mathbb{R} \text{ and } a \neq 0 \right\}$. Prove that H is a **subgroup** of $GL_2(\mathbb{R})$.

   Proof. We first choose $a = 1$ and $b = 0$. Then, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ belongs to H.
   Next, we will show that H is closed. Let $A = \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ 0 & \frac{1}{x} \end{bmatrix}$ be elements in H. Then

   $$AB = \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} x & y \\ 0 & \frac{1}{x} \end{bmatrix} = \begin{bmatrix} ax & ay + \frac{b}{x} \\ 0 & \frac{1}{ax} \end{bmatrix}$$

   Since $A, B \in H$, $a \neq 0$ and $x \neq 0$. We conclude that $ax \neq 0$. Thus, $AB \in H$.
   Finally, let $A = \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix}$ be in H. Then $a \neq 0$. Choose $A^{-1} = \begin{bmatrix} \frac{1}{a} & -b \\ 0 & a \end{bmatrix}$. Then

   $$AA^{-1} = \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} \frac{1}{a} & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & -b \\ 0 & a \end{bmatrix} \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} = A^{-1}A.$$

   Hence, A^{-1} is an inverse of A and belongs to H. Therefore, $H \leq GL_2(\mathbb{R})$. □

2. **(3 points)** Compute

   2.1 the **number of all generators** of $\mathbb{Z}_{86400}$

   $$\begin{aligned} \text{The number of all generators of } \mathbb{Z}_{43200} &= \phi(86400) \\ &= \phi(2^7 \cdot 3^3 \cdot 5^2) \\ &= \phi(2^7)\phi(3^3)\phi(5^2) \\ &= (2^7 - 2^6)(3^3 - 3^2)(5^2 - 5) \\ &= (64)(18)(20) \\ &= 23040 \quad \# \end{aligned}$$

   2.2 **index** of $[S_6 : \langle (1\,2\,3\,4)(4\,3\,5\,1) \rangle]$
   Consider

   $$(1\,2\,3\,4)(4\,3\,5\,1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} = (1\,2\,5)$$
   $$\circ((1\,2\,3\,4)(4\,3\,5\,1)) = 3$$

   Thus,

   $$[S_6 : \langle (1\,2\,3\,4)(4\,3\,5\,1) \rangle] = \frac{\circ(S_6)}{\circ((1\,2\,3\,4)(4\,3\,5\,1))} = \frac{6!}{3} = 240 \quad \#$$

3. **(4 points)** Find **all subgroups** and draw **lattice diagram** of the following groups

   3.1 $\mathbb{Z}_{24}$

   We choose 1 to be a generator of $\mathbb{Z}_{24}$. Then all divisors pf 24 are 1, 2, 3, 4, 6, 8, 12, 24. Hence, all subgroups are

$\langle 1 \rangle = \mathbb{Z}_{24}$

$\langle 2 \rangle = \{0, 2, 4, ..., 22\}$

$\langle 3 \rangle = \{0, 3, 6, ..., 21\}$

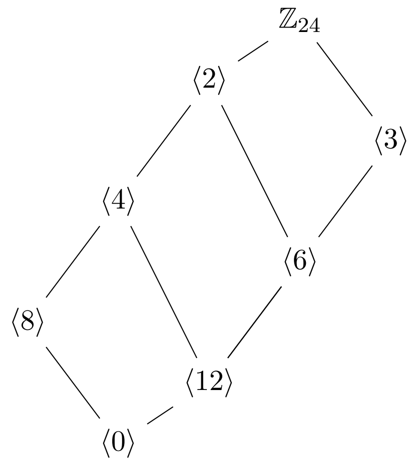
$\langle 4 \rangle = \{0, 4, 8, ..., 20\}$

$\langle 6 \rangle = \{0, 6, 12, 18\}$

$\langle 8 \rangle = \{0, 8, 16\}$

$\langle 12 \rangle = \{0, 12\}$

$\langle 0 \rangle = \{0\}$



3.2 $\mathbb{Z}_{27}^{\times}$

For $\mathbb{Z}_{27}^{\times} = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$, we consider

$2^0 = 1$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32 = 5$

$2^6 = 10$

$2^7 = 20$

$2^8 = 40 = 13$

$2^9 = 26$

$2^{10} = 52 = 25$

$2^{11} = 50 = 23$

$2^{12} = 46 = 19$

$2^{13} = 38 = 11$

$2^{14} = 22$

$2^{15} = 44 = 17$

$2^{16} = 34 = 7$

$2^{17} = 14$

$2^{18} = 28 = 1$

Hence, $\langle 2 \rangle = \mathbb{Z}_{27}^{\times}$. That is 2 to be a generator of $\mathbb{Z}_{27}^{\times}$. Since $\circ(\mathbb{Z}_{27}^{\times}) = 18$, all divisors of 18 are 1, 2, 3, 6, 9, 18. Then,

$\left\langle 2^{\frac{18}{1}} \right\rangle = \langle 2^{18} \rangle = \langle 1 \rangle = \{1\}$

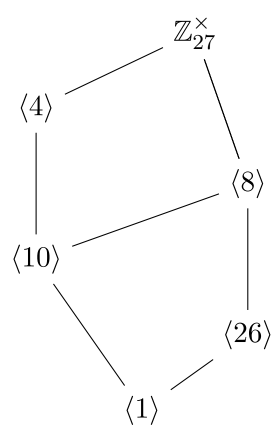
$\left\langle 2^{\frac{18}{2}} \right\rangle = \langle 2^{9} \rangle = \langle 26 \rangle = \{1, 26\}$

$\left\langle 2^{\frac{18}{3}} \right\rangle = \langle 2^{6} \rangle = \langle 10 \rangle = \{1, 10, 19\}$

$\left\langle 2^{\frac{18}{6}} \right\rangle = \langle 2^{3} \rangle = \langle 8 \rangle = \{1, 8, 10, 26, 19, 17\}$

$\left\langle 2^{\frac{18}{9}} \right\rangle = \langle 2^{2} \rangle = \langle 4 \rangle = \{1, 4, 16, 10, 13, 25, 19, 22, 7\}$

$\left\langle 2^{\frac{18}{18}} \right\rangle = \langle 2^{1} \rangle = \langle 2 \rangle = \mathbb{Z}_{27}^{\times}$



QUIZ 3 : MAT2303 ABSTRACT ALGEBRA

TOPIC Rings **SCORE** 10 points
QUIZ TIME Fri 3 Nov 2017, 11th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Determine $I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ is right ideal or left ideal of $M_{33}(\mathbb{R})$.
2. **(3 points)** Show that $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ is a subsring of $\mathbb{C}$. Show that $\mathbb{Z}[\sqrt{3}]$ has more than 4 elements which are units.
3. **(4 points)** Let $\varphi : \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\varphi(x) = \bar{x}^3$. Show that φ is a ring homomorphism, find $Ker(\varphi)$ and $Im(\varphi)$. Use the first isomorphism theorem to show that $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$

SOLUTION QUIZ 3 : MAT2303

TOPIC	Rings **SCORE** 10 points
QUIZ TIME	Fri 3 Nov 2017, 11th Week, Semester 1/2017
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Determine $I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ is right ideal or left ideal of $M_{33}(\mathbb{R})$.

Proof. Let $\begin{bmatrix} x & y & z \\ s & r & t \\ u & w & p \end{bmatrix} \in M_{33}(\mathbb{R})$ and $\begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} \in I$. Then

$$\begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ s & r & t \\ u & w & p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ ax+bs+cu & ay+br+cw & az+bt+cp \\ 0 & 0 & 0 \end{bmatrix} \in I$$

Thus, I is a right ideal of $M_{33}(\mathbb{R})$ but not a left ideal because

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \notin I$$

Hence, I is not an ideal of $M_{33}(\mathbb{R})$. □

2. **(3 points)** Show that $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ is a subsring of $\mathbb{C}$. Show that $\mathbb{Z}[\sqrt{3}]$ has more than 4 elements which are units.

Proof. Let $a + b\sqrt{3}, x + y\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ where $a, b, x, y \in \mathbb{Z}$. Since $a + x, b + y \in \mathbb{Z}$,

$$(a + b\sqrt{3}) + (x + y\sqrt{3}) = (a + x) + (b + y)\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$$

Since $0 = 0 + 0\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$, $0 \in \mathbb{Z}[\sqrt{3}]$. For each $a + b\sqrt{3}$, we get

$$(a + b\sqrt{3}) + (-a - b\sqrt{3}) = 0$$

and $-a - b\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$. Thus, $(\mathbb{Z}[\sqrt{3}], +)$ is an abelian group. It is obviously that $(\mathbb{Z}[\sqrt{3}], \cdot)$ is associative because $\mathbb{Z}[\sqrt{3} \subseteq \mathbb{R}$. Next, we will show that $(\mathbb{Z}[\sqrt{3}], \cdot)$ is closed. It is easy to see that

$$(a + b\sqrt{3})(x + y\sqrt{3}) = (ax + 3by) + (ay + bx)\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$$

□

Finally, let $x + y\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ be an inverse of $a + b\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ with $a \neq 0$ or $b \neq 0$, we have

$$(a + b\sqrt{3})(x + y\sqrt{3}) = 1$$
$$(ax + 3by) + (ay + bx)\sqrt{3} = 1 + 0\sqrt{3}$$

So, $ax + 3by = 1$ and $ay + bx = 0$. Suppose $a = 0$. Then $3by = 1$ is imposible. Thus, $a \neq 0$. Then $y = -\frac{bx}{a}$ and we get

$$ax + 3b\left(-\frac{bx}{a}\right) = 1$$
$$x\left(\frac{a^2 - 3b^2}{a}\right) = 1$$
$$x = \frac{a}{a^2 - 3b^2} \in \mathbb{Z} \quad \text{and} \quad y = -\frac{bx}{a} \in \mathbb{Z}$$

- If $a = \pm 1$, then $x = \frac{\pm 1}{1-3b^2}$. That is $1 - 3b^2 = \pm 1$. Then $b = 0$
- If $a = \pm 2$, then $x = \frac{\pm 2}{4-3b^2}$. That is $4 - 3b^2 = \pm 1, \pm 2$. Then $b = \pm 1$.

Thus,

Elements	Inverses
-1	-1
1	1
$2+\sqrt{3}$	$-2+\sqrt{3}$
$2-\sqrt{3}$	$-2-\sqrt{3}$
$-2+\sqrt{3}$	$2+\sqrt{3}$
$-2-\sqrt{3}$	$2-\sqrt{3}$

3. **(4 points)** Let $\varphi : \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\varphi(x) = \bar{x}^3$. Show that φ is a ring homomorphism, find $Ker(\varphi)$ and $Im(\varphi)$. Use the first isomorphism theorem to show that $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$

Proof. Let $x, y \in \mathbb{Z}$. Then

$$\begin{aligned}
\varphi(x+y) &= (\overline{x+y})^3 = (\bar{x}+\bar{y})^3 \\
&= \bar{x}^3 + 3\bar{x}^2\bar{y} + 3\bar{x}\bar{y}^2 + \bar{y}^3 = \bar{x}^3 + 0 + 0 + \bar{y}^3 \\
&= \bar{x}^3 + \bar{y}^3 = \varphi(x) + \varphi(y) \\
\varphi(xy) &= \overline{xy}^3 = (\bar{x}\bar{y})^3 = \bar{x}^3\bar{y}^3 \\
&= \varphi(x)\varphi(y)
\end{aligned}$$

Thus, φ is a ring homomorphism. Next, we find $Ker(\varphi)$ and $Im(\varphi)$. That is

$$\begin{aligned}
Ker(\varphi) &= \{x \in \mathbb{Z} : \varphi(x) = \bar{0}\} \\
&= \{x \in \mathbb{Z} : \bar{x}^3 = \bar{0}\} \\
&= 3\mathbb{Z} \\
Im(\varphi) &= \{y \in \mathbb{Z}_3 : \exists x \in \mathbb{Z}, \varphi(x) = y\} \\
&= \{y \in \mathbb{Z}_3 : \exists x \in \mathbb{Z}, \bar{x}^3 = y\} \\
&= \{0, 1, 2\} = \mathbb{Z}_3
\end{aligned}$$

By the first isomorphism theorem, $\mathbb{Z}/Ker(\varphi) \cong Im(\varphi)$. That is

$$\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$$

□

QUIZ 4 : MAT2303 ABSTRACT ALGEBRA

TOPIC Integral domain **SCORE** 10 points
QUIZ TIME Thur 9 Nov 2017, 13th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Find all maximal ideals and prime ideals of $\mathbb{Z}_{2310}$
2. **(3 points)** Find all irreducible elements in $\mathbb{Z}_{15}$
3. **(4 points)** Find all prime elements in $\mathbb{Z}_8$ by table

$\cdot$	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

SOLUTION QUIZ 4 : MAT2303 (SEC2)

TOPIC	Integral domain **SCORE** 10 points
QUIZ TIME	Thur 9 Nov 2017, 13th Week, Semester 1/2017
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Find all maximal ideals and prime ideals of $\mathbb{Z}_{2310}$
   Since $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, all prime divisors of 2310 are 2, 3, 5, 7, and 11. Thus, all maximal ideals and prime ideals are

$$(2), \quad (3), \quad (5), \quad (7), \quad \text{and} \quad (11)$$

2. **(3 points)** Find all irreducible elements in $\mathbb{Z}_{15}$
   All non units are 3, 5, 6, 9, 10 and 12.

·	3	5	6	9	10	12
3	9	0	3	12	0	6
5	0	10	0	0	5	0
6	3	0	6	9	0	12
9	6	10	9	6	0	3
10	0	5	0	0	10	0
12	9	0	12	3	0	9

   There is no irreducible elements in $\mathbb{Z}_{15}$.

3. **(4 points)** Find all prime elements in $\mathbb{Z}_8$ by table

·	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

   All non units are 2, 4 and 6.

   - Since $2 \mid x$ for $x = 0, 2, 4, 6$ (see 2 th row), if $2 \mid ab$, then $2 \mid a$ or $2 \mid b$. Thus, 2 is a prime element.
   - Since $4 = 2 \cdot 2$, $4 \mid (2 \cdot 2)$ but $4 \nmid 2$ (see 4th row), 4 is not a prime element.
   - Since $6 \mid x$ for $x = 0, 2, 4, 6$ (see 6 th row), if $6 \mid ab$, then $6 \mid a$ or $6 \mid b$. Thus, 6 is a prime element.

QUIZ 4 : MAT2303 ABSTRACT ALGEBRA

TOPIC Integral domain **SCORE** 10 points
QUIZ TIME Fri 10 Nov 2017, 13th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Find all maximal ideals and prime ideals of $\mathbb{Z}_{2418}$
2. **(3 points)** Find all irreducible elements in $\mathbb{Z}_{16}$
3. **(4 points)** Find all prime elements in $\mathbb{Z}_{10}$ by table

$\cdot$	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

SOLUTION QUIZ 4 : MAT2303 (SEC1)

TOPIC Integral domain **SCORE** 10 points
QUIZ TIME Fri 10 Nov 2017, 13th Week, Semester 1/2017
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Find all maximal ideals and prime ideals of $\mathbb{Z}_{2418}$
Since $2418 = 2 \cdot 3 \cdot 13 \cdot 31$, all prime divisors of 2418 are 2, 3, 13 and 31. Thus, all maximal ideals and prime ideals are

$$(2), \quad (3), \quad (13) \quad \text{and} \quad (31)$$

2. **(3 points)** Find all irreducible elements in $\mathbb{Z}_{16}$
All non unit of $\mathbb{Z}_{16}$ are 2, 4, 6, 8, 10, 12 and 14. Then

·	2	4	6	8	10	12	14
2	4	8	12	0	4	8	12
4	8	0	8	0	8	0	8
6	12	8	4	0	12	8	4
8	0	0	0	0	0	0	0
10	4	8	12	0	4	8	12
12	8	0	8	0	8	0	8
14	12	8	4	0	12	8	4

In the table, it appears 0, 4, 8 and 12. Hence, 2, 6, 10 and 14 are irreducible.

3. **(4 points)** Find all prime elements in $\mathbb{Z}_{10}$ by table

·	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	0	2	4	6	8
3	3	6	9	2	5	8	1	4	7
4	4	8	2	6	0	4	8	2	6
5	5	0	5	0	5	0	5	0	5
6	6	2	8	4	0	6	2	8	4
7	7	4	1	8	5	2	9	6	3
8	8	6	4	2	0	8	6	4	2
9	9	8	7	6	5	4	3	2	1

All non units are 2, 4, 5, 6 and 8.

- Since $2 \mid x$ for $x = 0, 2, 4, 6, 8$ (see 2 th row), if $2 \mid ab$, then $2 \mid a$ or $2 \mid b$. Thus, 2 is a prime element.
- Since $4 \mid x$ for $x = 0, 2, 4, 6, 8$ (see 4 th row), if $4 \mid ab$, then $4 \mid a$ or $4 \mid b$. Thus, 4 is a prime element.
- Since $5 \mid x$ for $x = 0, 5$ (see 5 th row), if $5 \mid ab$, then $5 \mid a$ or $5 \mid b$. Thus, 5 is a prime element.
- Since $6 \mid x$ for $x = 0, 2, 4, 6, 8$ (see 6 th row), if $6 \mid ab$, then $6 \mid a$ or $6 \mid b$. Thus, 6 is a prime element.
- Since $8 \mid x$ for $x = 0, 2, 4, 6, 8$ (see 8 th row), if $8 \mid ab$, then $8 \mid a$ or $8 \mid b$. Thus, 8 is a prime element.