QUIZ 1 : MAT2305 LINEAR ALGEBRA

TOPIC Linear equation system & Matrix **SCORE** 10 points
QUIZ TIME Wed 25 Jan 2017, 3th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**........................

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1. **(3 points)** Use Guass-Jardan elimination to solve linear equation systems

$$\begin{cases} x_1 + 2x_2 - 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= -1 \end{cases}$$

2. **(5 points)** Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

3. **(2 points)** Write linear system equation system in $A\boldsymbol{x} = \boldsymbol{b}$ and use inverse of A in 2 to solve the system.

$$\begin{cases} x_1 + 2x_4 &= 1 \\ x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 + 5x_3 &= 3 \\ x_3 + x_4 &= 4 \end{cases}$$

ANSWER QUIZ 1 : MAT2305 LINEAR ALGEBRA (SEC1)

TOPIC Linear equation system & Matrix **SCORE** 10 points
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TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
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1. **(3 points)** Use Guass-Jardan elimination to solve linear equation systems

$$\begin{cases} x_1 + 2x_2 - 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= -1 \end{cases}$$

Solution. Write out the linear equation system in augmented matrix to be

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 2 & 3 & 1 & 7 \\ 1 & 1 & 4 & -1 \end{array}\right]$$

Then

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 2 & 3 & 1 & 7 \\ 1 & 1 & 4 & -1 \end{array}\right] \xrightarrow[R_2-2R_1]{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & -1 & 7 & -9 \\ 0 & -1 & 7 & -9 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & -1 & 7 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & 3x_3 & = & 8 \\ & - & x_2 & + & 7x_3 & = & -9 \\ & & & & 0 & = & 0 \end{array}$$

Let t be a free variable. Setting $x_3 = t$. Then $x_2 = 9 + 7t$ and

$$x_1 = 8 - 2x_2 + 3x_3 = 8 - 2(9 + 7t) + 3t = -10 - 11t$$

Hence, $x_1 = -10 - 11t, x_2 = 9 + 7t, x_3 = t$ is the solution. #

2. **(5 points)** Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 5 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4-R_3]{R_2-3R_3}\left[\begin{array}{cccc|cccc}1&0&0&2&1&0&0&0\\0&1&0&-6&-3&-5&3&0\\0&0&1&2&1&2&-1&0\\0&0&0&-1&-1&-2&1&1\end{array}\right]$$

$$\xrightarrow{(-1)R_4}\left[\begin{array}{cccc|cccc}1&0&0&2&1&0&0&0\\0&1&0&-6&-3&-5&3&0\\0&0&1&2&1&2&-1&0\\0&0&0&1&1&2&-1&-1\end{array}\right]$$

$$\xrightarrow[R_3-2R_4,R_2+6R_4]{R_1-2R_4}\left[\begin{array}{cccc|cccc}1&0&0&0&-1&-4&2&2\\0&1&0&0&3&7&-3&-6\\0&0&1&0&-1&-2&1&2\\0&0&0&1&1&2&-1&-1\end{array}\right]$$

Thus,

$$A^{-1}=\begin{bmatrix}-1&-4&2&2\\3&7&-3&-6\\-1&-2&1&2\\1&2&-1&-1\end{bmatrix}\qquad \#$$

3. **(2 points)** Write linear system equation system in $A\boldsymbol{x}=\boldsymbol{b}$ and use inverse of A in 2 to solve the system.

$$\begin{cases}x_1+2x_4 &=1\\x_2+3x_3 &=2\\x_1+2x_2+5x_3 &=3\\x_3+x_4 &=4\end{cases}$$

Solution. We write linear system equation system in $A\boldsymbol{x}=\boldsymbol{b}$, i.e.,

$$\begin{bmatrix}1&0&0&2\\0&1&3&0\\1&2&5&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Taking A^{-1} to two left sides and By 2, we obtain

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1&0&0&2\\0&1&3&0\\1&2&5&0\\0&0&1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}-1&-4&2&2\\3&7&-3&-6\\-1&-2&1&2\\1&2&-1&-1\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}-1-8+6+8\\3+14-9-24\\-1-4+3+8\\1+4-3-4\end{bmatrix}=\begin{bmatrix}5\\-16\\6\\-2\end{bmatrix}$$

Therefore, $x_1=5, x_2=-16, x_3=6, x_4=-2$ is the solution. #

QUIZ 1 : MAT2305 LINEAR ALGEBRA

TOPIC Linear equation system & Matrix **SCORE** 10 points
QUIZ TIME Fri 27 Jan 2017, 3th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Use Guass-Jardan elimination to solve linear equation systems

$$\begin{cases} x_1 + 4x_2 + 2x_3 &= 6 \\ -x_1 - 3x_2 &= 1 \\ 2x_1 + 7x_2 + 2x_3 &= 5 \end{cases}$$

2. **(5 points)** Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

3. **(2 points)** Write linear system equation system in $A\boldsymbol{x} = \boldsymbol{b}$ and use inverse of A in 2 to solve the system.

$$\begin{cases} x_1 + x_3 &= 1 \\ x_2 - 2x_4 &= 2 \\ x_3 + x_4 &= 3 \\ x_1 + x_2 + 2x_3 &= 4 \end{cases}$$

ANSWER QUIZ 1 : MAT2305 LINEAR ALGEBRA (SEC2)

TOPIC Linear equation system & Matrix **SCORE** 10 points
QUIZ TIME Fri 27 Jan 2017, 3th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Use Guass-Jardan elimination to solve linear equation systems

$$\begin{cases} x_1 + 4x_2 + 2x_3 &= 6 \\ -x_1 - 3x_2 &= 1 \\ 2x_1 + 7x_2 + 2x_3 &= 5 \end{cases}$$

Solution. Write out the linear equation system in augmented matrix to be

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 6 \\ -1 & -3 & 0 & 1 \\ 2 & 7 & 2 & 5 \end{array}\right]$$

Then

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 6 \\ -1 & -3 & 0 & 1 \\ 2 & 7 & 2 & 5 \end{array}\right] \xrightarrow[R_2+R_1]{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -2 & -7 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$\begin{array}{ccccccc} x_1 & + & 4x_2 & + & 2x_3 & = & 6 \\ & & x_2 & + & 2x_3 & = & 7 \\ & & & & 0 & = & 0 \end{array}$$

Let t be a free variable. Setting $x_3 = t$. Then $x_2 = 7 - 2t$ and

$$x_1 = 6 - 4x_2 - 2x_3 = 6 - 4(7-2t) - 2t = -22 + 6t$$

Hence, $x_1 = -22 + 6t, x_2 = 7 - 2t, x_3 = t$ is the solution. #

2. **(5 points)** Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4-R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4-R_3]{R_1-R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+2R_4, R_3-R_4]{R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & -1 & -2 & 1 \\ -2 & -1 & -2 & 2 \\ 1 & 1 & 2 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad \#$$

3. **(2 points)** Write linear system equation system in $A\boldsymbol{x} = \boldsymbol{b}$ and use inverse of A in 2 to solve the system.

$$\begin{cases} x_1 + x_3 & = 1 \\ x_2 - 2x_4 & = 2 \\ x_3 + x_4 & = 3 \\ x_1 + x_2 + 2x_3 & = 4 \end{cases}$$

Solution. We write linear system equation system in $A\boldsymbol{x} = \boldsymbol{b}$, i.e.,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Taking A^{-1} to two left sides and By 2, we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & 1 \\ -2 & -1 & -2 & 2 \\ 1 & 1 & 2 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0-2-6+4 \\ -2-2-6+8 \\ 1+2+6-4 \\ -1-2-3+4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 5 \\ -2 \end{bmatrix}$$

Therefore, $x_1 = -4, x_2 = -2, x_3 = 5, x_4 = -2$ is the solution. #

QUIZ 2 : MAT2305 LINEAR ALGEBRA

TOPIC LU Decomposition & Determinant **SCORE** 10 points
QUIZ TIME Wed 8 Feb 2017, 5th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME... **ID**.................................... **SECTION**.......................

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1. **(3 points)** Use LU decomposition to solve linear equation systems

$$\begin{cases} x_1 + x_2 - x_3 & = 0 \\ 2x_1 - x_2 - x_3 & = -3 \\ x_1 - 2x_2 + x_3 & = 2 \end{cases}$$

2. **(3 points)** Find deteminant of A by EROs or cofactor expansion

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 3 \\ 3 & 4 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

3. **(4 points)** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} 2a_{21} & 2a_{22} \\ a_{11} & a_{12} \end{bmatrix}$. If $\det(A) = 2$, find

3.1 $\det(B)$

3.2 $\det(2A^2B)$

3.3 $\det(A^{-1}B(-A^2)^T)$

3.4 $\det(A+B)$

ANSWER QUIZ 2 : MAT2305 LINEAR ALGEBRA (SEC1)

TOPIC	LU Decomposition & Determinant **SCORE** 10 points
QUIZ TIME	Wed 8 Feb 2017, 5th Week, Semester 2/2016
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Use LU decomposition to solve linear equation systems

$$\begin{cases} x_1 + x_2 - x_3 & = 0 \\ 2x_1 - x_2 - x_3 & = -3 \\ x_1 - 2x_2 + x_3 & = 2 \end{cases}$$

Solution.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} = \boldsymbol{b}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[(-2)R_1+R_2]{(-1)R_1+R_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow{(-1)R_2+R_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[(+2)R_1+R_2]{(+1)R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{(+1)R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$$

$$A\boldsymbol{x} = LU\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

Setting $\boldsymbol{y} = U\boldsymbol{x}$, that is

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}$$

Then

$$L\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} = \boldsymbol{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \quad \leftrightarrow \quad \begin{array}{ccccccc} y_1 & & & & & = & 0 \\ 2y_1 & + & y_2 & & & = & -3 \\ y_1 & + & y_2 & + & y_3 & = & 2 \end{array}$$

$$\begin{aligned} & \rightarrow \quad \therefore \quad y_1 = 0 \\ 2y_1 + y_2 = -3 & \rightarrow \quad \therefore \quad y_2 = -3 - 2(0) = -3 \\ y_1 + y_2 + y_3 = 2 & \rightarrow \quad \therefore \quad y_3 = 2 - (-3) - 0 = 5 \end{aligned}$$

Since $\boldsymbol{y} = U\boldsymbol{x}$,

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \quad \leftrightarrow \quad \begin{matrix} x_1 & + & x_2 & - & x_3 & = & 0 \\ & - & 3x_2 & + & x_3 & = & -3 \\ & & & & x_3 & = & 5 \end{matrix}$$

$$\begin{aligned} & \rightarrow \quad \therefore \quad x_3 = 5 \\ -3x_2 + x_3 = -3 & \rightarrow \quad \therefore \quad x_2 = \frac{1}{-3}(-3-5) = \frac{8}{3} \\ x_1 + x_2 - x_3 = 0 & \rightarrow \quad \therefore \quad x_1 = 5 - \frac{8}{3} = \frac{7}{3} \end{aligned}$$

Verify your answer

$$\begin{cases} x_1 + x_2 - x_3 & = 0 \\ 2x_1 - x_2 - x_3 & = -3 \\ x_1 - 2x_2 + x_3 & = 2 \end{cases} \quad \rightarrow \quad \begin{cases} \frac{7}{3} + \frac{8}{3} - 5 & = 0 \\ 2(\frac{7}{3}) - \frac{8}{3} - 5 & = -3 \\ \frac{7}{3} - 2(\frac{8}{3}) + 5 & = 2 \end{cases}$$

Thus, $x_1 = \frac{7}{3}, x_2 = \frac{8}{3}, x_3 = 5$ is the solution. #

2. **(3 points)** Find deteminant of A by EROs or cofactor expansion

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 3 \\ 3 & 4 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution.

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 3 \\ 3 & 4 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \#$$

3. **(4 points)** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} 2a_{21} & 2a_{22} \\ a_{11} & a_{12} \end{bmatrix}$. If $\det(A) = 2$, find

Solution.

3.1 Since $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 2a_{21} & 2a_{22} \\ a_{11} & a_{12} \end{bmatrix} = B, \quad \det(B) = -4$ #

3.2 $\det(2A^2B) = 2^2\det(A^2)\det(B) = 4[\det(A)]^2\det(B) = 4(2)^2(-4) = -64$ #

3.3

$$\begin{aligned} \det(A^{-1}B(-A^2)^T) &= [\det(A)]^{-1}\det(B)\det(-A^2) = [\det(A)]^{-1}\det(B)(-1)^2[\det(A)]^2 \\ &= (2)^{-1}(-4)(2)^2 = -8 \quad \# \end{aligned}$$

3.4 $A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 2a_{21} & 2a_{22} \\ a_{11} & a_{12} \end{bmatrix} = \begin{bmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ a_{21} + a_{11} & a_{22} + a_{12} \end{bmatrix}$

$$\det(A+B) = \begin{vmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ a_{21} + a_{11} & a_{22} + a_{12} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{21} + a_{11} & a_{22} + a_{12} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = -2 \quad \#$$

QUIZ 2 : MAT2305 LINEAER ALGEBRA

TOPIC LU Decomposition & Determinant **SCORE** 10 points
QUIZ TIME Fri 10 Feb 2017, 5th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(3 points)** Use LU decomposition to solve linear equation systems

$$\begin{cases} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 - 3x_3 &= -4 \\ 5x_1 - x_2 + x_3 &= 6 \end{cases}$$

2. **(3 points)** Find deteminant of A by EROs or cofactor expansion

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 7 & 1 \\ 3 & 1 & 0 & 1 \\ 2 & 5 & 0 & -1 \end{bmatrix}$$

3. **(4 points)** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} + 2a_{21} & a_{12} + 2a_{22} \end{bmatrix}$. If $\det(B) = 4$, find

   3.1 $\det(A)$

   3.2 $\det(3A^T B^2)$

   3.3 $\det(-A^{-1}B(-2B)^T)$

   3.4 $\det(A + B)$

ANSWER QUIZ 2 : MAT2305 LINEAR ALGEBRA (SEC2)

TOPIC LU Decomposition & Determinant **SCORE** 10 points
QUIZ TIME Fri 10 Feb 2017, 5th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. **(3 points)** Use LU decomposition to solve linear equation systems

$$\begin{cases} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 - 3x_3 &= -4 \\ 5x_1 - x_2 + x_3 &= 6 \end{cases}$$

Solution.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix} \xrightarrow[(-1)R_1+R_2]{(-5)R_1+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{(+6)R_2+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -28 \end{bmatrix} = U$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[(+1)R_1+R_2]{(+5)R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \xrightarrow{(-6)R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} = L$$

$$A\boldsymbol{x} = LU\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 6 \end{bmatrix}$$

Setting $\boldsymbol{y} = U\boldsymbol{x}$, that is

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}$$

Then

$$L\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 6 \end{bmatrix} \quad \leftrightarrow \quad \begin{array}{rcrcrcr} y_1 & & & & & = & 6 \\ y_1 & + & y_2 & & & = & -4 \\ 5y_1 & - & 6y_2 & + & y_3 & = & 6 \end{array}$$

$$\begin{array}{rcll} & \to & \therefore & y_1 = 6 \\ y_1 + y_2 = -4 & \to & \therefore & y_2 = -4 - 6 = -10 \\ 5y_1 - 6y_2 + y_3 = 6 & \to & \therefore & y_3 = 6 - 5(6) + 6(-10) = -84 \end{array}$$

Since $\boldsymbol{y} = U\boldsymbol{x}$,

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ -84 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ -84 \end{bmatrix} \quad \leftrightarrow \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ & & x_2 & - & 4x_3 & = & -10 \\ & & & - & 28x_3 & = & -84 \end{array}$$

$$\begin{aligned} -28x_3 = -84 \quad &\rightarrow \quad \therefore \quad x_3 = 3 \\ x_2 - 4x_3 = -10 \quad &\rightarrow \quad \therefore \quad x_2 = -10 + 4(3) = 2 \\ x_1 + x_2 + x_3 = 6 \quad &\rightarrow \quad \therefore \quad x_1 = 6 - 3 - 2 = 1 \end{aligned}$$

Verify your answer

$$\begin{cases} x_1 + x_2 + x_3 & = 6 \\ x_1 + 2x_2 - 3x_3 & = -4 \\ 5x_1 - x_2 + x_3 & = 6 \end{cases} \quad \rightarrow \quad \begin{cases} 1 + 2 + 3 & = 6 \\ 1 + 2(2) - 3(3) & = -4 \\ 5(1) - 2 + 3 & = 6 \end{cases}$$

Thus, $x_1 = 1, x_2 = 2, x_3 = 3$ is the solution. #

2. **(3 points)** Find deteminant of A by EROs or cofactor expansion

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 7 & 1 \\ 3 & 1 & 0 & 1 \\ 2 & 5 & 0 & -1 \end{bmatrix}$$

Solution.

$$\det(A) = \begin{vmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 7 & 1 \\ 3 & 1 & 0 & 1 \\ 2 & 5 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 9 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & 5 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 9 & 5 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -43 & -22 \end{vmatrix} = 6 \begin{vmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 9 & 5 \\ 0 & 0 & -1 & \frac{1}{3} \\ 0 & 0 & -43 & -22 \end{vmatrix}$$

$$= 6 \begin{vmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 9 & 5 \\ 0 & 0 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{109}{3} \end{vmatrix} = 6(-1)(1)(-1)(-\frac{109}{3}) = -218 \quad \#$$

3. **(4 points)** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} + 2a_{21} & a_{12} + 2a_{22} \end{bmatrix}$. If $\det(B) = 4$, find

3.1 Since $B = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} + 2a_{21} & a_{12} + 2a_{22} \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} a_{11} & a_{12} \\ 2a_{21} & 2a_{22} \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$, $\det(A) = 2$ #

3.2 $\det(3A^T B^2) = 3^2 \det(A)[\det(B)]^2 = 9(2)(4)^2 = 288$ #

3.3

$$\begin{aligned} \det(-A^{-1}B(-2B)^T) &= \det(-A^{-1})\det(B)\det(-2B)^T = (-1)^2[\det(A)]^{-1}\det(B)\det(-2B) \\ &= (2)^{-1}(4)(-2)^2\det(B) = 8(4) = 32 \quad \# \end{aligned}$$

3.4 $A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{11} + 2a_{21} & a_{12} + 2a_{22} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} \\ a_{11} + 3a_{21} & a_{12} + 3a_{22} \end{bmatrix}$

$$\det(A + B) = \begin{vmatrix} 2a_{11} & 2a_{12} \\ a_{11} + 3a_{21} & a_{12} + 3a_{22} \end{vmatrix} = 2 \begin{vmatrix} a_{11} & a_{12} \\ a_{11} + 3a_{21} & a_{12} + 3a_{22} \end{vmatrix} = 2 \begin{vmatrix} a_{11} & a_{12} \\ 3a_{21} & 3a_{22} \end{vmatrix}$$

$$= 2(3) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 6\det(A) = 6(2) = 12 \quad \#$$

QUIZ 3 : MAT2305 LINEAR ALGEBRA

TOPIC Linear Transformation **SCORE** 10 points
QUIZ TIME Wed 22 Mar 2017, 11th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(4 points)** Let $B = \{(-1,1,0),(1,1,0),(0,0,1)\}$ be a basis for $\mathbb{R}^3$ and T be a linear operator on $\mathbb{R}^3$. If

$$T(-1,1,0) = (1,2,3) \quad T(1,1,0) = (3,2,1) \quad T(0,2,1) = (3,3,3)$$

   Find fomular of $T(x,y,z)$

2. **(4 points)** Let T be a linear operator on $\mathbb{R}^4$ such that

$$T(x,y,z,w) = (x+2y+w, -x-2y-w, 3x+5y+z+3w, -y+z)$$

   Compute the nullity and rank of T.

3. **(2 points)** Find the inverse (if it exists) of T where $T(x,y) = (x+y, 2x+y)$.

QUIZ 3 : MAT2305 LINEAR ALGEBRA

TOPIC Linear Transformation **SCORE** 10 points
QUIZ TIME Thur 23 Mar 2017, 11th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. **(4 points)** Let $B = \{(1,-1,0),(1,1,0),(0,0,-1)\}$ be a basis for $\mathbb{R}^3$ and T be a linear operator on $\mathbb{R}^3$. If

$$T(1,-1,0) = (1,2,3) \quad T(1,1,0) = (3,2,1) \quad T(2,0,-1) = (4,4,4)$$

   Find fomular of $T(x,y,z)$

2. **(4 points)** Let T be a linear operator on $\mathbb{R}^4$ such that

$$T(x,y,z,w) = (-x+2z+w, 2x-4z-2w, -x+y+5z+3w, y+3z+2w)$$

   Compute the nullity and rank of T.

3. **(2 points)** Find the inverse (if it exists) of T where $T(x,y) = (3x+2y, 2x+y)$.

ANSWERS QUIZ 3 : MAT2305 LINEAR ALGEBRA (SEC1)

TOPIC Linear Transformation **SCORE** 10 points
QUIZ TIME Wed 22 Mar 2017, 11th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University

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1. **(4 points)** Let $B = \{(-1,1,0),(1,1,0),(0,0,1)\}$ be a basis for $\mathbb{R}^3$ and T be a linear operator on $\mathbb{R}^3$. If

$$T(-1,1,0) = (1,2,3) \quad T(1,1,0) = (3,2,1) \quad T(0,2,1) = (3,3,3)$$

Find fomular of $T(x,y,z)$
Solution. Let $x,y,z \in \mathbb{R}$ and

$$\begin{aligned}
c_1(-1,1,0) + c_2(1,1,0) + c_3(0,2,1) &= (x,y,z) \\
(-c_1 + c_2, c_1 + c_2 + 2c_3, c_3) &= (x,y,z) \\
-c_1 + c_2 &= x \\
c_1 + c_2 + 2c_3 &= y \\
c_3 &= z
\end{aligned}$$

Then

$$\begin{aligned}
(-c_1 + c_2) + (c_1 + c_2 + 2c_3) &= x + y \\
2c_2 + 2c_3 &= x + y \\
2c_2 &= x + y - 2z \\
\therefore c_2 &= \frac{1}{2}(x + y - 2z) \\
c_1 = c_2 - x &= \frac{1}{2}(x + y - 2z) - x \\
\therefore c_1 &= \frac{1}{2}(y - x - 2z)
\end{aligned}$$

Thus,

$$\begin{aligned}
T(x,y,z) &= c_1T(-1,1,0) + c_2T(1,1,0) + c_3T(0,2,1) \\
T(x,y,z) &= c_1(1,2,3) + c_2(3,2,1) + c_3(3,3,3) \\
&= (c_1 + 3c_2 + 3c_3, 2c_1 + 2c_2 + 3c_3, 3c_1 + c_2 + 3c_3) \\
&= \left(\frac{1}{2}(y - x - 2z) + \frac{3}{2}(x + y - 2z) + 3z, (y - x - 2z) + (x + y - 2z) + 3z, \frac{3}{2}(y - x - 2z) + \frac{1}{2}(x + y - 2z) + 3z\right) \\
&= (x + 2y - z, 2y - z, -x + 2y - z) \quad \#
\end{aligned}$$

2. **(4 points)** Let T be a linear operator on $\mathbb{R}^4$ such that

$$T(x,y,z,w) = (x+2y+w, -x-2y-w, 3x+5y+z+3w, -y+z)$$

Compute the nullity and rank of T.
Solution. Consider,

$$T(x,y,z,w) = (x+2y+w, -x-2y-w, 3x+5y+z+3w, -y+z) = (0,0,0,0)$$

$$\begin{cases} x+2y+w & =0 \\ -x-2y-w & =0 \\ 3x+5y+z+3w & =0 \\ -y+z & =0 \end{cases} \leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ -1 & -2 & 0 & -1 & 0 \\ 3 & 5 & 1 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ -1 & -2 & 0 & -1 & 0 \\ 3 & 5 & 1 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_4-R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x+2y+w=0$ and $-y+z=0$. Set $z=t$ and $w=s$. Then $y=t$ and $x=-2t-s$. Thus

$$\begin{aligned} Ker(T) &= \{(x,y,z,w) : T(x,y,z,w) = (0,0,0,0)\} \\ &= \{(-2t-s, t, t, s) : t, s \in \mathbb{R}\} \\ &= \{t(-2,1,1,0) + s(-1,0,0,1) : t, s \in \mathbb{R}\} \\ &= Span\{(-2,1,1,0), (-1,0,0,1)\} \end{aligned}$$

Therefore, Nullity$(T) = 2$ and Rank$(T) = 4 - 2 = 2$ #

3. **(2 points)** Find the inverse (if it exists) of T where $T(x,y) = (x+y, 2x+y)$.
Solution. Consider, $T(x,y) = (0,0)$. So,

$$\begin{cases} x+y & =0 \\ 2x+y & =0 \end{cases}$$

Then, $0 = (2x+y)-(x+y) = x$ and $y = 0$. Thus, $Ker(T) = \{(x,y) : T(x,y) = (0,0)\} = \{(0,0)\}$. Hence, T is one-to-one. Setting

$$\begin{cases} x+y & =a \\ 2x+y & =b \end{cases}$$

Then, $b-a = (2x+y)-(x+y) = x$ and $y = a-(b-a) = 2a-b$. Thus,

$$\begin{aligned} T^{-1}(x+y, 2x+y) &= (x,y) \\ T^{-1}(a,b) &= (b-a, 2a-b) \\ T^{-1}(x,y) &= (y-x, 2x-y) \quad \# \end{aligned}$$

ANSWERS QUIZ 3: MAT2305 LINEAR AL (SEC2)

TOPIC	Linear Transformation	**SCORE**	10 points
QUIZ TIME	Thur 23 Mar 2017, 11th Week, Semester 2/2016		
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University		

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1. **(4 points)** Let $B = \{(1,-1,0),(1,1,0),(0,0,-1)\}$ be a basis for $\mathbb{R}^3$ and T be a linear operator on $\mathbb{R}^3$. If

$$T(1,-1,0) = (1,2,3) \quad T(1,1,0) = (3,2,1) \quad T(2,0,-1) = (4,4,4)$$

Find fomular of $T(x,y,z)$

Solution. Let $x,y,z \in \mathbb{R}$ and

$$\begin{aligned} c_1(1,-1,0) + c_2(1,1,0) + c_3(2,0,-1) &= (x,y,z) \\ (c_1+c_2+2c_3, -c_1+c_2, -c_3) &= (x,y,z) \\ c_1+c_2+2c_3 &= x \\ -c_1+c_2 &= y \\ -c_3 &= z \quad \therefore c_3 = -z \end{aligned}$$

Then

$$\begin{aligned} (c_1+c_2+2c_3) + (-c_1+c_2) &= x+y \\ 2c_2+2c_3 &= x+y \\ 2c_2 &= x+y+2z \\ \therefore c_2 &= \frac{1}{2}(x+y+2z) \\ c_1 = c_2 - y &= \frac{1}{2}(x+y+2z) - y \\ \therefore c_1 &= \frac{1}{2}(x-y+2z) \end{aligned}$$

Thus,

$$\begin{aligned} T(x,y,z) &= c_1T(1,-1,0) + c_2T(1,1,0) + c_3T(2,0,-1) \\ T(x,y,z) &= c_1(1,2,3) + c_2(3,2,1) + c_3(4,4,4) \\ &= (c_1+3c_2+4c_3, 2c_1+2c_2+4c_3, 3c_1+c_2+4c_3) \\ &= \left(\frac{1}{2}(x-y+2z) + \frac{3}{2}(x+y+2z) - 4z, (x-y+2z)+(x+y+2z)-4z, \frac{3}{2}(x-y+2z) + \frac{1}{2}(x+y+2z) - 4z\right) \\ &= (2x+y+z, 2x, 2x-y) \quad \# \end{aligned}$$

2. **(4 points)** Let T be a linear operator on $\mathbb{R}^4$ such that

$$T(x, y, z, w) = (-x + 2z + w, 2x - 4z - 2w, -x + y + 5z + 3w, y + 3z + 2w)$$

Compute the nullity and rank of T.

Solution. Consider,

$$T(x, y, z, w) = (-x + 2z + w, 2x - 4z - 2w, -x + y + 5z + 3w, y + 3z + w) = (0, 0, 0, 0)$$

$$\begin{cases} -x + 2z + w &= 0 \\ 2x - 4z - 2w &= 0 \\ -x + y + 5z + 3w &= 0 \\ y + 3z + 2w &= 0 \end{cases} \leftrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 2 & 1 & 0 \\ 2 & 0 & -4 & -2 & 0 \\ -1 & 1 & 5 & 3 & 0 \\ 0 & 1 & 3 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} -1 & 0 & 2 & 1 & 0 \\ 2 & 0 & -4 & -2 & 0 \\ -1 & 1 & 5 & 3 & 0 \\ 0 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_2+2R_1]{R_2-R_1} \left[\begin{array}{cccc|c} -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_4-R_3} \left[\begin{array}{cccc|c} -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, $y + 3z + 2w = 0$ and $-x + 2z + w = 0$. Set $z = t$ and $w = s$. Then $y = -3t - 2s$ and $x = 2t + s$. Thus

$$\begin{aligned} Ker(T) &= \{(x, y, z, w) : T(x, y, z, w) = (0, 0, 0, 0)\} \\ &= \{(2t + s, -3t - 2s, t, s) : t, s \in \mathbb{R}\} \\ &= \{t(2, -3, 1, 0) + s(1, -2, 0, 1) : t, s \in \mathbb{R}\} \\ &= Span\{(2, -3, 1, 0), (1, -2, 0, 1)\} \end{aligned}$$

Therefore, Nullity(T) = 2 and Rank(T) = $4 - 2 = 2$ #

3. **(2 points)** Find the inverse (if it exists) of T where $T(x, y) = (3x + 2y, 2x + y)$.
Solution. Consider, $T(x, y) = (0, 0)$. So,

$$\begin{cases} 3x + 2y &= 0 \\ 2x + y &= 0 \end{cases}$$

Then, $0 = 2(2x + y) - (3x + 2y) = x$ and $y = 0$. Thus, $Ker(T) = \{(x, y) : T(x, y) = (0, 0)\} = \{(0, 0)\}$. Hence, T is one-to-one. Setting

$$\begin{cases} 3x + 2y &= a \\ 2x + y &= b \end{cases}$$

Then, $2b - a = 2(2x + y) - (3x + 2y) = x$ and $y = b - 2(2b - a) = 2a - 3b$. Thus,

$$\begin{aligned} T^{-1}(x + y, 2x + y) &= (x, y) \\ T^{-1}(a, b) &= (2b - a, 2a - 3b) \\ T^{-1}(x, y) &= (2y - x, 2x - 3y) \quad \# \end{aligned}$$

QUIZ 4 : MAT2305 LINEAER ALGEBRA

TOPIC Matrix transformation & Inner product **SCORE** 10 points
QUIZ TIME Fri 7 Apr 2017, 13th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education,
Suan Sunandha Rajabhat University
NAME.. **ID**.................................... **SECTION**.......................

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1. (**4 points**) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

   Find $[T]_B$ where $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

2. (**3 points**) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product vector space V. Suppose

$$\|2\boldsymbol{u} - 3\boldsymbol{v}\| = \|3\boldsymbol{u} - 2\boldsymbol{v}\| = 1 \text{ and } \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1$$

   Compute $\|3\boldsymbol{u} + 4\boldsymbol{v}\|$

3. (**3 points**) Let $B = \{(1, 2, 2), (-2, 1, 0), (-2, -4, 5)\}$ be a basis for $\mathbb{R}^3$.

   3.1 Show that B is a orthogonal basis for $\mathbb{R}^3$

   3.2 Find $[\boldsymbol{v}]_B$ where $\boldsymbol{v} = (1, 3, 5)$

ANSWERS QUIZ 4 : MAT2305 LINEAER ALGEBRA (SEC2)

TOPIC Matrix transformation & Inner product **SCORE** 10 points
QUIZ TIME Fri 7 Apr 2017, 13th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University

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1. (**4 points**) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

Find $[T]_B$ where $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

Solution. Then,

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Next, we will find $c's$ constants.

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 1 & 1 & -1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 2 & 1 & -1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 & -1 & 1 \\ 0 & -2 & -1 & 0 & -2 & -1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & -2 & -1 & 0 & -2 & -1 \end{array}\right]$$
$$\xrightarrow[R_1-2R_2]{R_3+2R_2} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -3 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 3 & 0 & 0 & -3 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -3 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$
$$\xrightarrow[R_2-2R_3]{R_1+3R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

Thus,

$$[T]_B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \quad \#$$

2. (**3 points**) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product vector space V. Suppose

$$\|2\boldsymbol{u}-3\boldsymbol{v}\| = \|3\boldsymbol{u}-2\boldsymbol{v}\| = 1 \text{ and } \langle \boldsymbol{u},\boldsymbol{v}\rangle = 1$$

Compute $\|3\boldsymbol{u}+4\boldsymbol{v}\|$
Solution.

$$\begin{aligned}
\|2\boldsymbol{u}-3\boldsymbol{v}\|^2 &= \|3\boldsymbol{u}-2\boldsymbol{v}\|^2\\
\langle 2\boldsymbol{u}-3\boldsymbol{v}, 2\boldsymbol{u}-3\boldsymbol{v}\rangle &= \langle 3\boldsymbol{u}-2\boldsymbol{v}, 3\boldsymbol{u}-2\boldsymbol{v}\rangle\\
4\langle \boldsymbol{u},\boldsymbol{u}\rangle - 6\langle \boldsymbol{u},\boldsymbol{v}\rangle - 6\langle \boldsymbol{v},\boldsymbol{u}\rangle + 9\langle \boldsymbol{v},\boldsymbol{v}\rangle &= 9\langle \boldsymbol{u},\boldsymbol{u}\rangle - 6\langle \boldsymbol{u},\boldsymbol{v}\rangle - 6\langle \boldsymbol{v},\boldsymbol{u}\rangle + 4\langle \boldsymbol{v},\boldsymbol{v}\rangle\\
4\|\boldsymbol{u}\|^2 - 12\langle \boldsymbol{u},\boldsymbol{v}\rangle + 9\|\boldsymbol{v}\|^2 &= 9\|\boldsymbol{u}\|^2 - 12\langle \boldsymbol{u},\boldsymbol{v}\rangle + 4\|\boldsymbol{v}\|^2\\
5\|\boldsymbol{u}\|^2 &= 5\|\boldsymbol{v}\|^2\\
\|\boldsymbol{u}\| &= \|\boldsymbol{v}\|
\end{aligned}$$

Since $\|2\boldsymbol{u}-3\boldsymbol{v}\| = 1$ and $\langle \boldsymbol{u},\boldsymbol{v}\rangle = 1$,

$$\begin{aligned}
\|2\boldsymbol{u}-3\boldsymbol{v}\|^2 &= 1\\
4\|\boldsymbol{u}\|^2 - 12\langle \boldsymbol{u},\boldsymbol{v}\rangle + 9\|\boldsymbol{v}\|^2 &= 1\\
4\|\boldsymbol{u}\|^2 - 12(1) + 9\|\boldsymbol{u}\|^2 &= 1\\
13\|\boldsymbol{u}\|^2 - 12 &= 1\\
\therefore \|\boldsymbol{u}\| &= 1 \qquad \text{and} \qquad \|\boldsymbol{v}\| = 1
\end{aligned}$$

Thus,

$$\begin{aligned}
\|3\boldsymbol{u}+4\boldsymbol{v}\|^2 &= \langle 3\boldsymbol{u}+4\boldsymbol{v}, 3\boldsymbol{u}+4\boldsymbol{v}\rangle\\
&= 9\langle \boldsymbol{u},\boldsymbol{u}\rangle + 12\langle \boldsymbol{u},\boldsymbol{v}\rangle + 12\langle \boldsymbol{v},\boldsymbol{u}\rangle + 16\langle \boldsymbol{v},\boldsymbol{v}\rangle\\
&= 9\|\boldsymbol{u}\|^2 + 24\langle \boldsymbol{u},\boldsymbol{v}\rangle + 16\|\boldsymbol{v}\|^2\\
&= 9(1)^2 + 24(1) + 16(1)^2 = 49\\
\|3\boldsymbol{u}+4\boldsymbol{v}\| &= 7 \quad \#
\end{aligned}$$

3. (**3 points**) Let $B = \{(1,2,2), (-2,1,0), (-2,-4,5)\}$ be a basis for $\mathbb{R}^3$.

   3.1 Show that B is a orthogonal basis for $\mathbb{R}^3$
   Solution.

$$\begin{aligned}
\langle (1,2,2), (-2,1,0)\rangle &= 1(-2) + 2(1) + 2(0) = 0\\
\langle (1,2,2), (-2,-4,5)\rangle &= 1(-2) + 2(-4) + 2(5) = 0\\
\langle (-2,-4,5), (-2,1,0)\rangle &= -2(-2) - 4(1) + 2(5) = 0
\end{aligned}$$

   Thus, B is orthogonal basis for $\mathbb{R}^3$.

   3.2 Find $[\boldsymbol{v}]_B$ where $\boldsymbol{v} = (1,3,5)$
   Solution.

$$\begin{aligned}
c_1 &= \frac{\langle (1,3,5), (1,2,2)\rangle}{\|(1,2,2)\|^2} = \frac{1+6+10}{9} = \frac{17}{9}\\
c_2 &= \frac{\langle (1,3,5), (-2,1,0)\rangle}{\|(-2,1,0)\|^2} = \frac{-2+3+0}{5} = \frac{1}{5}\\
c_3 &= \frac{\langle (1,3,5), (-2,-4,5)\rangle}{\|(-2,-4,5)\|^2} = \frac{-2-12+25}{45} = \frac{11}{45}
\end{aligned}$$

   Therefore,

$$[v]_B = \left(\frac{17}{9}, \frac{1}{5}, \frac{11}{45}\right) \quad \#$$

QUIZ 4 : MAT2305 LINEAER ALGEBRA

TOPIC Matrix transformation & Inner product **SCORE** 10 points
QUIZ TIME Mon 10 Apr 2017, 13th Week, Semester 2/2016
TEACHER Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University
NAME.. **ID**..................................... **SECTION**........................

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1. (**4 points**) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_2 - x_3 \end{bmatrix}$$

Find $[T]_B$ where $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

2. (**3 points**) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product vector space V. Suppose

$$\|4\boldsymbol{u} - 3\boldsymbol{v}\| = \|3\boldsymbol{u} - 4\boldsymbol{v}\| = 1 \text{ and } \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1$$

Compute $\|2\boldsymbol{u} + 3\boldsymbol{v}\|$

3. (**3 points**) Let $B = \{(1, 2, 3), (3, -3, 1), (11, 8, -9)\}$ be a basis for $\mathbb{R}^3$.

   3.1 Show that B is a orthogonal basis for $\mathbb{R}^3$

   3.2 Find $[\boldsymbol{v}]_B$ where $\boldsymbol{v} = (1, 0, 1)$

ANSWERS QUIZ 4 : MAT2305 LINEAER ALGEBRA (SEC1)

TOPIC	Matrix transformation & Inner product	**SCORE**	10 points
QUIZ TIME	Mon 10 Apr 2017, 13th Week, Semester 2/2016		
TEACHER	Thanatyod Jampawai, Ph.D., Faculty of Education, Suan Sunandha Rajabhat University		

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1. (**4 points**) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_2 - x_3 \end{bmatrix}$$

Find $[T]_B$ where $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

Solution. Then,

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Next, we will find $c's$ constants.

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & -1 & -2 & 0 & 1 & -1 \\ 0 & -2 & -1 & 0 & 0 & -3 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & -2 & -1 & 0 & 0 & -3 \end{array}\right]$$
$$\xrightarrow[R_1 - 2R_2]{R_3 + 2R_2} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -3 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 3 & 0 & -2 & -1 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -3 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right]$$
$$\xrightarrow[R_2 - 2R_3]{R_1 + 3R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

Thus,

$$[T]_B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \quad \#$$

2. (**3 points**) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product vector space V. Suppose

$$\|4\boldsymbol{u} - 3\boldsymbol{v}\| = \|3\boldsymbol{u} - 4\boldsymbol{v}\| = 1 \text{ and } \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1$$

Compute $\|2\boldsymbol{u} + 3\boldsymbol{v}\|$
Solution.

$$\begin{aligned}
\|4\boldsymbol{u} - 3\boldsymbol{v}\|^2 &= \|3\boldsymbol{u} - 4\boldsymbol{v}\|^2 \\
\langle 4\boldsymbol{u} - 3\boldsymbol{v}, 4\boldsymbol{u} - 3\boldsymbol{v} \rangle &= \langle 3\boldsymbol{u} - 4\boldsymbol{v}, 3\boldsymbol{u} - 4\boldsymbol{v} \rangle \\
16\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 12\langle \boldsymbol{u}, \boldsymbol{v} \rangle - 12\langle \boldsymbol{v}, \boldsymbol{u} \rangle + 9\langle \boldsymbol{v}, \boldsymbol{v} \rangle &= 9\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 12\langle \boldsymbol{u}, \boldsymbol{v} \rangle - 12\langle \boldsymbol{v}, \boldsymbol{u} \rangle + 16\langle \boldsymbol{v}, \boldsymbol{v} \rangle \\
16\|\boldsymbol{u}\|^2 - 24\langle \boldsymbol{u}, \boldsymbol{v} \rangle + 9\|\boldsymbol{v}\|^2 &= 9\|\boldsymbol{u}\|^2 - 24\langle \boldsymbol{u}, \boldsymbol{v} \rangle + 16\|\boldsymbol{v}\|^2 \\
7\|\boldsymbol{u}\|^2 &= 7\|\boldsymbol{v}\|^2 \\
\|\boldsymbol{u}\| &= \|\boldsymbol{v}\|
\end{aligned}$$

Since $\|4\boldsymbol{u} - 3\boldsymbol{v}\| = 1$ and $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1$,

$$\begin{aligned}
\|4\boldsymbol{u} - 3\boldsymbol{v}\|^2 &= 1 \\
16\|\boldsymbol{u}\|^2 - 24\langle \boldsymbol{u}, \boldsymbol{v} \rangle + 9\|\boldsymbol{v}\|^2 &= 1 \\
16\|\boldsymbol{u}\|^2 - 24(1) + 9\|\boldsymbol{u}\|^2 &= 1 \\
25\|\boldsymbol{u}\|^2 - 24 &= 1 \\
\therefore \|\boldsymbol{u}\| &= 1 \quad \text{and} \quad \|\boldsymbol{v}\| = 1
\end{aligned}$$

Thus,

$$\begin{aligned}
\|2\boldsymbol{u} + 3\boldsymbol{v}\|^2 &= \langle 2\boldsymbol{u} + 3\boldsymbol{v}, 2\boldsymbol{u} + 3\boldsymbol{v} \rangle \\
&= 4\langle \boldsymbol{u}, \boldsymbol{u} \rangle + 6\langle \boldsymbol{u}, \boldsymbol{v} \rangle + 6\langle \boldsymbol{v}, \boldsymbol{u} \rangle + 9\langle \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= 4\|\boldsymbol{u}\|^2 + 12\langle \boldsymbol{u}, \boldsymbol{v} \rangle + 9\|\boldsymbol{v}\|^2 \\
&= 9(1)^2 + 12(1) + 16(1)^2 = 37 \\
\|2\boldsymbol{u} + 3\boldsymbol{v}\| &= \sqrt{37} \quad \#
\end{aligned}$$

3. (**3 points**) Let $B = \{(1,2,3), (3,-3,1), (11,8,-9)\}$ be a basis for $\mathbb{R}^3$.

   3.1 Show that B is a orthogonal basis for $\mathbb{R}^3$
   Solution.

$$\begin{aligned}
\langle (1,2,3), (3,-3,1) \rangle &= 1(3) + 2(-3) + 3(1) = 0 \\
\langle (1,2,3), (11,8,-9) \rangle &= 1(11) + 2(8) + 3(-9) = 0 \\
\langle (3,-3,1), (11,8,-9) \rangle &= 3(11) - 3(8) + 1(-9) = 0
\end{aligned}$$

   Thus, B is orthogonal basis for $\mathbb{R}^3$.

   3.2 Find $[\boldsymbol{v}]_B$ where $\boldsymbol{v} = (1,0,1)$
   Solution.

$$\begin{aligned}
c_1 &= \frac{\langle (1,0,1), (1,2,3) \rangle}{\|(1,2,3)\|^2} = \frac{1+0+3}{14} = \frac{2}{7} \\
c_2 &= \frac{\langle (1,0,1), (3,-3,1) \rangle}{\|(3,-3,1)\|^2} = \frac{3+0+1}{19} = \frac{4}{19} \\
c_3 &= \frac{\langle (1,0,1), (11,8,-9) \rangle}{\|(11,8,-9)\|^2} = \frac{11+0-9}{266} = \frac{1}{133}
\end{aligned}$$

   Therefore,

$$[v]_B = \left(\frac{2}{7}, \frac{4}{19}, \frac{1}{133}\right) \quad \#$$