Quiz 1
MAC3309 Mathematical Analysis

Topic	Ordered field axiom, Supremum & Infimum	**Score**	10 marks
Time	30 minutes ($3th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

Name .... **ID** .... **Sec** ....

---

1. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that
$$\sqrt{x^2+y^2} \le |x| + |y|.$$

2. **(5 marks)** Let $A = \left\{1 - \frac{2}{n} : n \in \mathbb{N}\right\}$. Find $\inf A$ and $\sup A$ with proving them.

Math SSRU

Solution Quiz 1
MAC3309 Mathematical Analysis

Topic	Ordered field axiom, Supremum & Infimume	**Score**	10 marks
Time	30 minutes ($3th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let $x, y \in \mathbb{R}$. Prove that
$$\sqrt{x^2+y^2} \leq |x| + |y|.$$

Solution. Let $x, y \in \mathbb{R}$. By the fact that $|x| \geq 0$ and $|y| \geq 0$, we have $2|x||y| \geq 0$. Then,
$$|x|^2 + 2|x||y| + |y|^2 \geq |x|^2 + |y|^2$$
$$(|x| + |y|)^2 \geq x^2 + y^2.$$
It follows that $|x| + |y| \geq \sqrt{x^2+y^2}$.

2. **(5 marks)** Let $A = \left\{1 - \dfrac{2}{n} : n \in \mathbb{N}\right\}$. Find $\inf A$ and $\sup A$ with proving them.

$A = \left\{-1, 0, \dfrac{1}{3}, \dfrac{1}{2}, ...\right\}$. **Claim that** $\inf A = -1$ and $\sup A = 1$

Proof. $\inf A = -1$
Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $-2n \leq -2$. We obatin
$$-2 \leq -\frac{2}{n}$$
$$-1 = -2 + 1 \leq 1 - \frac{2}{n}.$$
Thus, -1 is a lower bound of A.
Let ℓ be a lower bound of A. For $n = 1$, we get $-1 \in A$. So, $\ell \leq -1$. Hence, $\inf A = -1$.
$\sup A = 1$
Let $n \in \mathbb{N}$. Then $n \geq 0$. So, $-\dfrac{2}{n} < 0$. We have
$$1 - \frac{2}{n} < 1.$$
Thus, 1 is an upper bound of A.
Assume that that there is an upper bound u_0 of A such that
$$u_0 < 1.$$
By definition,
$$1 - \frac{2}{n} \leq u_0 \quad \text{for all } n \in \mathbb{N} \qquad (*)$$
From $\dfrac{1-u_0}{2} > 0$. By Archimendean property, there is an $n_0 \in \mathbb{N}$ such that
$$\frac{1}{n_0} < \frac{1-u_0}{2}$$
$$\frac{2}{n_0} < 1 - u_0$$
$$u_0 < 1 - \frac{2}{n_0}$$
This is contradiction to $(*)$. Therefore, $\sup A = 1$. ∎

Quiz 2
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	30 minutes ($5th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

Name .... **ID** .... **Sec** ....

---

1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}.$$

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{n^2}{n+2} = +\infty.$$

Solution Quiz 2
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	30 minutes (5*th* Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}.$$

Proof. Let $\varepsilon > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{1}{n} \leq \frac{1}{N}$. Since $4n+6 > 4n$, $\frac{1}{4n+6} < \frac{1}{4n}$. Hence,

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| = \left|\frac{2n-(2n+3)}{2(2n+3)}\right| = \frac{3}{4n+6} < \frac{3}{4n} < \frac{4}{4n} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$. □

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{n^2}{n+2} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $M+2 < N$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n-2 > N-2$. Since $0 > -4$, $n^2 > n^2 - 4$. We obtain

$$\frac{n^2}{n+2} > \frac{n^2-4}{n+2} = \frac{(n-2)(n+2)}{n+2} = n-2 > N-2 > M.$$

Hence, $\lim_{n\to\infty} \frac{n^2}{n+2} = +\infty$. □

Quiz 3
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

Name .... **ID** .... **Sec** ....

---

1. **(5 marks)** Let $f(x) = x^2 + x - 1$. Use the Definition to prove that

$$f \text{ is continuous at } 1.$$

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\arctan x \leq x \quad \text{ for all } x \geq 0.$$

Solution Quiz 3
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let $f(x) = x^2 + x - 1$. Use the Definition to prove that

$$f \text{ is continuous at } 1.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$ such that $|x - 1| < \delta$. Then $|x - 1| < 1$.

$$\text{So, } |x| - |2| \leq |x + 2| < 1. \text{ We obtain } |x| \leq 3.$$

It follows that

$$\begin{aligned} |f(x) - f(1)| &= \left|(x^2 + x - 1) - 1\right| = \left|x^2 + x - 2\right| \\ &= |(x - 1)(x + 2)| = |x - 1||x + 2| \\ &< \delta(|x| + 2) < \delta(3 + 2) < \frac{\varepsilon}{5} \cdot 5 = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 1$. □

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\arctan x \leq x \quad \text{for all } x \geq 0.$$

Proof. Let $a > 0$ and $f(x) = \arctan x - x$ on $[0, a]$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. Then, $f'(x) = \dfrac{1}{1 + x^2} - 1$. By the Mean Value Theorem (MVT), there is a $c \in (0, a)$ such that

$$\begin{aligned} f(a) - f(0) &= f'(c)(a - 0) \\ (\arctan a - a) - (0 - 0) &= \left(\frac{1}{1 + c^2} - 1\right) a \\ \arctan a - a &= \left(\frac{-c^2}{1 + c^2}\right) a \end{aligned}$$

Since $1 + c^2 > 0$ and $-c^2 < 0$, $\dfrac{-c^2}{1 + c^2} < 0$. So, $\left(\dfrac{-c^2}{1 + c^2}\right) a \leq 0$ because $a > 0$.
Therefore,

$$\arctan x \leq x \quad \text{for all } x \geq 0.$$

□

Quiz 4
MAC3309 Mathematical Analysis

Topic	Integrable & Riemann sum	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

Name .... **ID** .... **Sec** ....

---

1. **(5 marks)** Let

$$f(x) = \begin{cases} 2 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x < 2. \end{cases}$$

Show that f is **integrable** on $[0, 2]$

2. **(5 marks)** Let $f(x) = x^2$ where $x \in [0, 2]$ and

$$P = \left\{ \frac{2j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0, 2]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution Quiz 4
MAC3309 Mathematical Analysis

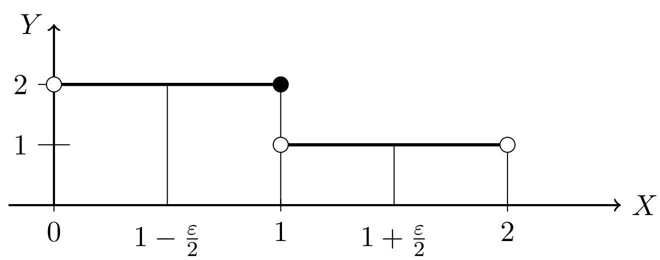
Topic	Integrable & Riemann sum	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2022
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let
$$f(x) = \begin{cases} 2 & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x < 2. \end{cases}$$

Show that f is **integrable** on $[0,2]$

Solution. Let $\varepsilon > 0$. Case $\varepsilon \le 1$. Choose $P = \left\{0, 1-\frac{\varepsilon}{2}, 1, 1+\frac{\varepsilon}{2}, 2\right\}$.



We obtain

$$\begin{aligned} U(f,P) &= 2\left(1-\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 1\left(1-\frac{\varepsilon}{2}\right) \\ L(f,P) &= 2\left(1-\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(1-\frac{\varepsilon}{2}\right) \\ U(f,P) - L(f,P) &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Case $\varepsilon > 1$. Choose $P = \{0,1,2\}$. Then

$$\begin{aligned} U(f,P) &= 2(1-0) + 2(2-1) \\ L(f,P) &= 2(1-0) + 1(2-1) \\ U(f,P) - L(f,P) &= 1 < \varepsilon. \end{aligned}$$

Thus, f is integrable on $[0,2]$.

2. **(5 marks)** Let $f(x) = x^2$ where $x \in [0, 2]$ and

$$P = \left\{ \frac{2j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0, 2]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \frac{2j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{2}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= f\left(\frac{2}{n}\right)\frac{2}{n} + f\left(\frac{4}{n}\right)\frac{2}{n} + f\left(\frac{6}{n}\right)\frac{2}{n} + \cdots + f\left(\frac{2n}{n}\right)\frac{2}{n} \\
&= \frac{2}{n}\left[\frac{2^2}{n^2} + \frac{4^2}{n^2} + \frac{6^2}{n^2} + \cdots + \frac{(2n)^2}{n^2}\right] \\
&= \frac{2}{n^3}\left[2^2 + 4^2 + 6^2 + \cdots + (2n)^2\right] \\
&= \frac{2}{n^3}\left[2^2(1^2 + 2^2 + 3^2 + \cdots + n^2)\right] \\
&= \frac{8}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] \\
&= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{4(n+1)(2n+1)}{3n^2}
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{4(n+1)(2n+1)}{3n^2} = \frac{8}{3} \quad \#$$