Math SSRU

Quiz 1 : (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Ordered field axiom, Supremum & Infimum	**Score**	10 marks
Time	30 minutes (3*th* Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $x \in \mathbb{R}$ such that $0 < x < 1$. Prove that

$$x < \sqrt{x}.$$

2. **(5 marks)** Let $A = \left\{ \dfrac{2}{n+1} : n \in \mathbb{N} \right\}$. Find **inf** $\boldsymbol{A}$ and prove it.

Math SSRU

Solution Quiz 1 : (8 a.m.)
MAC3309 Mathematical Analysis

Topic Ordered field axiom, Supremum & Infimume **Score** 10 marks
Time 30 minutes ($3th$ Week) **Semester** 2/2023
Teacher Assistant Professor Thanatyod Jampawai, Ph.D.
Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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1. **(5 marks)** Let $x \in \mathbb{R}$ such that $0 < x < 1$. Prove that

$$x < \sqrt{x}.$$

Proof. Let $x \in \mathbb{R}$ such that $0 < x < 1$. Then $x > 0$. By O3.1, we have

$$x^2 = x \cdot x < 1 \cdot x = x.$$

We obtain $x^2 - x < 0$. It follows that

$$x^2 - (\sqrt{x})^2 < 0$$
$$(x - \sqrt{x})(x + \sqrt{x}) < 0.$$

Since $x + \sqrt{x} > 0$, $(x + \sqrt{x})^{-1} > 0$. By O3.1 again,

$$(x - \sqrt{x})(x + \sqrt{x})(x + \sqrt{x})^{-1} < 0 \cdot (x + \sqrt{x})^{-1}$$
$$x - \sqrt{x} < 0.$$

We conclude that $x < \sqrt{x}$.

□

2. **(5 marks)** Let $A = \left\{ \frac{2}{n+1} : n \in \mathbb{N} \right\}$. Find **inf *A*** and prove it.

We see that $A = \left\{ 1, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \ldots \right\}$. **Claim that** $\inf A = 0$.

Proof. We will prove that $\inf A = 0$
Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $n + 1 > 0$. We obtain

$$\frac{2}{n+1} > 0$$

Thus, 0 is a lower bound of A.
Finally, we will show that 0 is the greatest lower bound of A.
Assume that that there is a lower bound ℓ_0 of A such that

$$\ell_0 > 0.$$

By definition,

$$\ell_0 \leq \frac{2}{n+1} \quad \text{for all } n \in \mathbb{N} \qquad (*)$$

From $\frac{\ell_0}{2} > 0$. By Archimendean property (2), there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\ell_0}{2} \qquad \longrightarrow \qquad \frac{2}{n_0} < \ell_0$$

Since $n_0 + 1 > n_0$,

$$\frac{2}{n_0 + 1} < \frac{2}{n_0} < \ell_0$$

This is contradiction to $(*)$. Therefore, $\inf A = 0$.

□

Quiz 1 : (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Ordered field axiom, Supremum & Infimum	**Score**	10 marks
Time	30 minutes ($3th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $x, y \in \mathbb{R}^+$. Prove that

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2.$$

2. **(5 marks)** Let $A = \left\{ \frac{2n}{n+1} : n \in \mathbb{N} \right\}$. Find **sup *A*** and prove it.

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Solution Quiz 1 : (1 p.m.)
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	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $x, y \in \mathbb{R}^+$. Prove that
$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2.$$

Proof. Let $x, y \in \mathbb{R}^+$. In the fact that $(\sqrt{x} - \sqrt{y})^2 \geq 0$, we obtain

$$\begin{aligned} x - 2\sqrt{x}\sqrt{y} + y &\geq 0 \\ x + y &\geq 2\sqrt{x}\sqrt{y} \\ \frac{x}{\sqrt{x}\sqrt{y}} + \frac{y}{\sqrt{x}\sqrt{y}} &\geq 2 \\ \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} &\geq 2. \end{aligned}$$

□

2. **(5 marks)** Let $A = \left\{ \frac{2n}{n+1} : n \in \mathbb{N} \right\}$. Find **sup *A*** and prove it.

We see that $A = \left\{ 1, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, ... \right\}$. **Claim that** $\sup A = 2$.

Proof. We will prove that $\sup A = 2$
Let $n \in \mathbb{N}$. Then $n \geq 1$. From $0 < 2$ So, $0 + 2n < 2 + 2n$. We obtain

$$\begin{aligned} 2n &< 2(n+1) \\ \frac{2n}{n+1} &< 2 \end{aligned}$$

Thus, 2 is an upper bound of A.
Finally, we will show that 2 is the least upper bound of A.
Assume that that there is an upper bound u_0 of A such that

$$u_0 < 2.$$

By definition,

$$\frac{2n}{n+1} \leq u_0 \quad \text{for all } n \in \mathbb{N} \qquad (*)$$

From $\frac{2 - u_0}{2} > 0$. By Archimendean property (2), there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2 - u_0}{2} \qquad \longrightarrow \qquad \frac{2}{n_0} < 2 - u_0$$

Since $n_0 + 1 > n_0$,

$$\begin{aligned} \frac{2}{n_0+1} < \frac{2}{n_0} &< 2 - u_0 \\ u_0 &< 2 - \frac{2}{n_0+1} = \frac{2n_0}{n_0+1}. \end{aligned}$$

This is contradiction to $(*)$. Therefore, $\sup A = 2$. $\square$

Quiz 2 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	30 minutes ($5th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2.$$

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n^2}{n+1} = +\infty.$$

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Solution Quiz 2 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
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Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
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1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2.$$

Proof. Let $\varepsilon > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{1}{n} \leq \frac{1}{N}$. Since $n+1 > n$, $\frac{1}{n+1} < \frac{1}{n}$. Hence,

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n - 2(n+1)}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{2n}{n+1} = \frac{1}{2}$. □

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n^2}{n+1} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that

$$N > \frac{M+2}{2}.$$

It's equivalent to $2N - 2 > M$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Then $2n - 2 > 2N - 2$. Since $0 > -2$, $2n^2 > 2n^2 - 2$. We obtain

$$\frac{2n^2}{n+1} > \frac{2n^2 - 2}{n+1} = \frac{2(n-1)(n+1)}{n+1} = 2n - 2 > 2N - 2 > M.$$

Hence, $\lim_{n\to\infty} \frac{2n^2}{n+1} = +\infty$. □

Quiz 2 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	30 minutes ($5th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0.$$

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{1-n^2}{n} = -\infty.$$

Solution Quiz 2 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Limit of Sequences	**Score**	10 marks
Time	30 minutes (5*th* Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
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1. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0.$$

Proof. Let $\varepsilon > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{2}{n} \leq \frac{2}{N}$. Since $n^2+1 > n^2$, $\frac{1}{n^2+1} < \frac{1}{n^2}$. Hence,

$$\left|\frac{2n}{n^2+1} - 0\right| = \frac{2n}{n^2+1} = \frac{2n}{n^2} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$. □

2. **(5 marks)** Use the Definition to prove that

$$\lim_{n\to\infty} \frac{1-n^2}{n} = -\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that

$$N > 1 - M.$$

It's equivalent to $1 - N < M$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Then $-n \leq -N$. So, $1 - n \leq 1 - N$
Since $1 \leq n$, $1 - n^2 \leq n - n^2$. We obtain

$$\frac{1-n^2}{n} \leq \frac{n-n^2}{n} = \frac{n(1-n)}{n} = 1 - n \leq 1 - N < M.$$

Hence, $\lim_{n\to\infty} \frac{1-n^2}{n} = -\infty$. □

Quiz 3 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 2.$$

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x \leq x - 1 \quad \text{for all } x \geq 1.$$

Hints : Let $a > 1$ and consider a defined function on $[1, a]$.

Solution Quiz 3 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes (11*th* Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 2.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{24}\}$ such that $|x-2| < \delta$. Then $|x-2| < 1$.

$$\text{So, } |x| - |2| \leq |x-2| < 1. \text{ We obtain } |x| \leq 3.$$

By triangle inequility, it follows that

$$\begin{aligned} |f(x) - f(2)| &= |(x-1)(x-2)(x-3) - 0| \\ &= |x-1||x-2||x-3| \\ &< (|x|+1)\delta(|x|+3) \\ &< (3+1)\delta(3+3) \\ &= 24\delta < 24 \cdot \frac{\varepsilon}{24} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 2$. □

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x \leq x - 1 \quad \text{for all } x \geq 1.$$

Hints : Let $a > 1$ and consider function on $[1, a]$.

Proof. Let $a > 1$ and $f(x) = \ln x - x$ on $[1, a]$. Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. Then, $f'(x) = \dfrac{1}{x} - 1$. By the Mean Value Theorem (MVT), there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a-1) \\ (\ln a - a) - (0 - 1) &= \left(\frac{1}{c} - 1\right)(a-1) \\ (\ln a - a) + 1 &= \left(\frac{1-c}{c}\right)(a-1) \end{aligned}$$

From $1 < c < a$, $1 - c < 0$ and $a - 1 > 0$, we obtain

$$\left(\frac{1-c}{c}\right)(a-1) < 0.$$

So, $\ln a - a + 1 < 0$. Therefore,

$$\ln x \leq x - 1 \quad \text{for all } x \geq 0.$$

□

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Quiz 3 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 3.$$

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x < x \quad \text{ for all } x \geq 1.$$

Hints : Let $a > 1$ and consider a defined function on $[1, a]$.

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Solution Quiz 3 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2023
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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 3.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{30}\}$ such that $|x-3| < \delta$. Then $|x-3| < 1$.

$$\text{So, } |x| - |3| \leq |x-3| < 1. \text{ We obtain } |x| \leq 4.$$

By triangle inequility, it follows that

$$\begin{aligned} |f(x) - f(3)| &= |(x-1)(x-2)(x-3) - 0| \\ &= |x-1||x-2||x-3| \\ &< (|x|+1)(|x|+2|)\delta \\ &< (4+1)(4+2)\delta \\ &= 30\delta < 30 \cdot \frac{\varepsilon}{30} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 3$. □

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x < x \quad \text{for all } x \geq 1.$$

Hints : Let $a > 1$ and consider function on $[1, a]$.

Proof. Let $a > 1$ and $f(x) = \ln x - x$ on $[1, a]$. Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. Then, $f'(x) = \dfrac{1}{x} - 1$. By the Mean Value Theorem (MVT), there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a-1) \\ (\ln a - a) - (0-1) &= \left(\frac{1}{c} - 1\right)(a-1) \\ (\ln a - a) + 1 &= \left(\frac{1-c}{c}\right)(a-1) \end{aligned}$$

From $1 < c < a$, $1 - c < 0$ and $a - 1 > 0$, we obtain

$$\left(\frac{1-c}{c}\right)(a-1) < 0.$$

So, $\ln a - a + 1 < 0$. It follows that $\ln a - a < -1 < 0$.
Therefore,

$$\ln x < x \quad \text{for all } x \geq 0.$$

□

Quiz 4 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes (13*th* Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = 6x(x-1)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\int_{-1}^{0} f(x)\,dx = 67$. Use the change variable to compute

$$\int_{1}^{e} f(x \ln x - x) \cdot \ln x^2 \, dx.$$

Solution Quiz 4 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes (13th Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = 6x(x-1)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \dfrac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \dfrac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be

$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 6 \cdot \frac{j}{n}\left(\frac{j}{n} - 1\right) \\ &= \frac{6}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2} - \frac{j}{n}\right) = \frac{6}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 - \frac{1}{n}\sum_{j=1}^{n} j\right] \\ &= \frac{6}{n}\left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] \\ &= \frac{(n+1)(2n+1)}{n^2} - \frac{3(n+1)}{n}\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} - \frac{3(n+1)}{n} = 2 - 3 = -1 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\displaystyle\int_{-1}^{0} f(x)\,dx = 67$. Use the change variable to compute

$$\int_1^e f(x\ln x - x) \cdot \ln x^2\,dx.$$

Solution. Let $\phi(x) = x\ln x - x$. Then $\phi'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x$,

$$\phi(1) = 1\ln 1 - 1 = 0 - 1 = -1 \quad \text{and} \quad \phi(e) = e\ln e - e = e - e = 0.$$

By the change variable, we obtain

$$\begin{aligned}\int_1^e f(x\ln x - x) \cdot \ln x^2\,dx &= \int_1^e f(\phi(x)) \cdot 2\ln x\,dx \\ &= 2\int_1^e f(\phi(x)) \cdot \phi'(x)\,dx \\ &= 2\int_{\phi(1)}^{\phi(e)} f(t)\,dt \\ &= 2\int_{-1}^{0} f(t)\,dt = 5 \cdot 67 = 134 \quad \#\end{aligned}$$

Quiz 4 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes (13th Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = 3x(x+2)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\int_0^1 f(x)\,dx = 67$. Use the change variable to compute

$$\int_0^1 f(e^x - xe^x) \cdot xe^x \, dx.$$

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Solution Quiz 4 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
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1. **(5 marks)** Let $f(x) = 3x(x+2)$ where $x \in [0,1]$ and
$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$
be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \frac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be
$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 3 \cdot \frac{j}{n}\left(\frac{j}{n} + 2\right) \\ &= \frac{3}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2} + 2 \cdot \frac{j}{n}\right) = \frac{3}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 + \frac{2}{n}\sum_{j=1}^{n} j\right] \\ &= \frac{3}{n}\left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{2}{n} \cdot \frac{n(n+1)}{2}\right] \\ &= \frac{(n+1)(2n+1)}{2n^2} + \frac{3(n+1)}{n}\end{aligned}$$
Thus,
$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{2n^2} + \frac{3(n+1)}{n} = 1 + 3 = 4 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\int_0^1 f(x)\,dx = 67$. Use the change variable to compute
$$\int_0^1 f(e^x - xe^x) \cdot xe^x\,dx.$$

Solution. Let $\phi(x) = e^x - xe^x$. Then $\phi'(x) = e^x - (x \cdot e^x + 1 \cdot e^x) = -xe^x$,
$$\phi(0) = e^0 - 0e^0 = 1 = 1 - 0 = 1 \quad \text{and} \quad \phi(1) = e - e = 0.$$
By the change variable, we obtain
$$\begin{aligned}\int_0^1 f(e^x - xe^x) \cdot xe^x\,dx &= -\int_0^1 f(\phi(x)) \cdot (-xe^x)\,dx \\ &= -\int_0^1 f(\phi(x)) \cdot \phi'(x)\,dx \\ &= -\int_{\phi(0)}^{\phi(1)} f(t)\,dt \\ &= -\int_1^0 f(t)\,dt = \int_0^1 f(t)\,dt = 67 \quad \#\end{aligned}$$

Quiz 4 (Addition)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
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Name .... **ID** .... **Sec** ....

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1. **(5 marks)** Let $f(x) = 6(x-1)(x+1)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

   be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\displaystyle\int_1^{1+e} f(x)\,dx = 66$. Use the change variable to compute

$$\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{2x}\,dx.$$

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Solution Quiz 4 (Addition)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2023
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1. **(5 marks)** Let $f(x) = 6(x-1)(x+1)$ where $x \in [0,1]$ and
$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$
be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \dfrac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \dfrac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be
$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 6\left(\frac{j}{n}-1\right)\left(\frac{j}{n}+1\right) \\ &= \frac{6}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2}-1\right) = \frac{6}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} 1\right] \\ &= \frac{6}{n}\left[\frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} - n\right] \\ &= \frac{(n+1)(2n+1)}{n^2} - 6\end{aligned}$$
Thus,
$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2} - 6 = 2 - 6 = -4 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\displaystyle\int_1^{1+e} f(x)\,dx = 66$. Use the change variable to compute
$$\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{2x}\,dx.$$

Solution. Let $\phi(x) = \ln(xe^x) = \ln x + \ln e^x = \ln x + x$. Then $\phi'(x) = \frac{1}{x} + 1 = \frac{1+x}{x}$,
$$\phi(1) = \ln 1 + 1 = 0 + 1 = 1 \quad \text{and} \quad \phi(e) = \ln e + e = 1 + e.$$
By the change variable, we obtain
$$\begin{aligned}\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{2x}\,dx &= \frac{1}{2}\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{x}\,dx \\ &= \frac{1}{2}\int_1^e f(\phi(x)) \cdot \phi'(x)\,dx \\ &= \frac{1}{2}\int_{\phi(1)}^{\phi(e)} f(t)\,dt \\ &= \frac{1}{2}\int_1^{1+e} f(t)\,dt = \frac{1}{2}\cdot 66 = 33 \quad \#\end{aligned}$$