MATHEMATICAL ANALYSIS

Division of Mathematics Faculty of Education

Suan Sunandha Rajabhat University

2022

MATHEMATICAL ANALYSIS

Thanatyod Jampawai
Assistant Professor of Mathematics
Suan Sunandha Rajabhat University

Division of Mathematics, Faculty of Education
Suan Sunandha Rajabhat University
Bangkok, Thailand
Update : November 2022

Contents

Chapter 1

The Real Number System

1.1 Ordered field axioms

FIELD AXIOMS.

There are functions $+$ and $\cdot$, defined on $\mathbb{R}^2$, that satisfy the following properties for every $a, b, c \in \mathbb{R}$:

F1	**Closure Properties**	$a+b$ and $a \cdot b$ belong to $\mathbb{R}$.
F2	**Associative Properties**	$a+(b+c)=(a+b)+c$
		$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
F3	**Commutative Properties**	$a+b=b+a$ and $a \cdot b = b \cdot a$
F4	**Distributive Properties**	$a \cdot (b+c) = a \cdot b + a \cdot c$
		$(b+c) \cdot a = b \cdot a + c \cdot a$
F5	**Additive Identity**	There is a unique element $0 \in \mathbb{R}$ such that
		$0+a=a=a+0$ for all $a \in \mathbb{R}$.
F6	**Multiplicative Identity**	There is a unique element $1 \in \mathbb{R}$ such that
		$1 \cdot a = a = a \cdot 1$ for all $a \in \mathbb{R}$.
F7	**Additive Inverse**	For every $x \in \mathbb{R}$ there is a unique $-x \in \mathbb{R}$ such that
		$x+(-x)=0=(-x)+x$.
F8	**Multiplicative Inverse**	For every $x \in \mathbb{R}\backslash\{0\}$ there is a unique $x^{-1} \in \mathbb{R}$ such that
		$x \cdot (x^{-1}) = 1 = (x^{-1}) \cdot x$.

We shall frequently denote

$$a+(-b) \text{ by } a-b, \quad a \cdot b \text{ by } ab, \quad a^{-1} \text{ by } \frac{1}{a} \quad \text{and} \quad a \cdot b^{-1} \text{ by } \frac{a}{b}.$$

The real number system $\mathbb{R}$ contains certain special subsets: the set of **natural numbers**

$$\mathbb{N} := \{1, 2, 3, ...\}$$

obtained by begining with 1 and successively adding 1's to form $2 := 1 + 1$, $3 := 2 + 1$, etc,; the set of **integers**

$$\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$$

(Zahlen is German for number); the set of **rationals** (or fractions or quoteints)

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

and the set of **irrationals**

$$\mathbb{Q}^c := \mathbb{R} \backslash \mathbb{Q}.$$

Equality in $\mathbb{Q}$ is defined by

$$\frac{m}{n} = \frac{p}{q} \quad \text{if and only if} \quad mq = np.$$

Recall that each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ is a proper subset of the next; i.e.,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Definition 1.1.1 *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Define*

$$a^n = \underbrace{a \cdot a \cdot ... \cdot a}_{n-\ copies}$$

*a and n are called **base** and **exponent**, respectively.*

Definition 1.1.2 *Let a be a non-zero real number. Define*

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n} \quad \text{for } n \in \mathbb{N}$$

Theorem 1.1.3 *Let $a, b \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Then*

1. $(ab)^n = a^n b^n$
2. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ *where* $b \neq 0$
3. $a^n \cdot a^m = a^{m+n}$
4. $\frac{a^n}{a^m} = a^{n-m}$ *where* $a \neq 0$

Proof. Excercise. □

Theorem 1.1.4 *Let a be a real number. Then*

1. $0a = 0$
2. $(-1)a = -a$
3. $-(-a) = a$
4. $(a^{-1})^{-1} = a$ *where* $a \neq 0$

Proof. Let a be a real number. We first consider

$$\begin{aligned} 0a &= (0+0)a && \text{(by F5)} \\ &= 0a + 0a && \text{(by F4)} \end{aligned}$$

By F5, it implies that $0a = 0$. This result leds to

$$\begin{aligned} 0 &= 0a && \text{(by 1.)} \\ &= (1 + (-1))a && \text{(by F7)} \\ &= 1a + (-1)a && \text{(by F4)} \\ &= a + (-1)a && \text{(by F6)} \end{aligned}$$

By F7, $(-1)a$ is an additive inverse of a. Thus, $(-1)a = -a$. This result leds to

$$0 = a + (-a)$$

So, a is an inverse of $-a$. Thus, $a = -(-a)$. For $a \neq 0$, by F8, we give

$$aa^{-1} = 1$$

Then, a is a multiplicative inverse of a^{-1}. So, $a = (a^{-1})^{-1}$. □

Theorem 1.1.5 *Let a and b be real numbers. Then*

$$-(ab) = a(-b) = (-a)b.$$

Proof. Let a and b be real numbers. We consider

$$\begin{aligned} 0 &= 0b && \text{(by 1. in Theorem 1.1.4)} \\ &= (a + (-a))b && \text{(by F7)} \\ &= ab + (-a)b && \text{(by F4)} \end{aligned}$$

Then, $(-a)b$ is an additive inverse of ab. So, $(-a)b = -(ab)$.

Similary, we will show that $a(-b) = -(ab)$. □

Theorem 1.1.6 (**Cancellation law**) *Let a, b and c be real numbers. Then*

1. *Cancellation law for addition*   *if* $a + c = b + c$, *then* $a = b$.
2. *Cancellation law for multiplication*   *if* $ac = bc$ *and* $c \neq 0$, *then* $a = b$.

Proof. Let a, b and c be real numbers. Assume that $a + c = b + c$. Then,

$$\begin{aligned}
a &= a + 0 && \text{(by F5)}\\
&= a + (c + (-c)) && \text{(by F7)}\\
&= (a + c) + (-c) && \text{(by F2)}\\
&= (b + c) + (-c) && \text{(by assumption)}\\
&= b + (c + (-c)) && \text{(by F2)}\\
&= b + 0 && \text{(by F7)}\\
&= b && \text{(by F5)}
\end{aligned}$$

Next, we assume that $ac = bc$ and $c \neq 0$. Then $c^{-1} \in \mathbb{R}$. We obtain

$$\begin{aligned}
a &= a1 && \text{(by F6)}\\
&= a(cc^{-1}) && \text{(by F8)}\\
&= (ac)c^{-1} && \text{(by F2)}\\
&= (bc)c^{-1} && \text{(by assumption)}\\
&= b(cc^{-1}) && \text{(by F2)}\\
&= b1 && \text{(by F8)}\\
&= b && \text{(by F6)}
\end{aligned}$$

□

Theorem 1.1.7 (**Integral Domain**) *Let a and b be real numbers.*

$$\text{If} \quad ab = 0 \ , \text{ then} \quad a = 0 \quad \text{or} \quad b = 0.$$

Proof. Let a and b be real numbers. Suppose $ab = 0$ and $a \neq 0$. By 1. in Theorem 1.1.4, we get

$$ab = 0 = a0$$

By cancellation for multiplication, $b = 0$. □

ORDER AXIOMS.

There is a relation $<$ on $\mathbb{R}^2$ that has the following properties for every $a, b, c \in \mathbb{R}$.

O1	**Trichotomy Property**	Given $a, b \in \mathbb{R}$, one and only one of the following statements holds: $a < b$, $b < a$, or $a = b$
O2	**Trasitive Property**	$a < b$ and $b < c$ imply $a < c$
O3	**Additive Property**	$a < b$ imply $a + c < b + c$
O4	**Multiplicative Property**	O4.1 $a < b$ and $0 < c$ imply $ac < bc$
		O4.2 $a < b$ and $c < 0$ imply $bc < ac$

We define in other cases:

- By $b > a$ we shall mean $a < b$.
- By $a \leq b$ we shall mean $a < b$ or $a = b$.
- If $a < b$ and $b < c$, we shall write $a < b < c$.
- We shall call a number $a \in \mathbb{R}$ **nonnegative** if $a \geq 0$ and **positive** if $a > 0$.

Example 1.1.8 *Let $x \in \mathbb{R}$. Show that if $0 < x < 1$, then $0 < x^2 < x$*

Proof. Let x be a real number such that $0 < x < 1$. Then $0 < x$ and $x < 1$. By O4.1 and the fact that $x > 0$, we obtain

$$0 = 0 \cdot x < x \cdot x = x^2 \quad \text{and} \quad x^2 = x \cdot x < 1 \cdot x = x$$

By O2, it implies that

$$0 \ < \ x^2 \ < \ x.$$

□

Example 1.1.9 *Let $x, y \in \mathbb{R}$. Show that if $0 < x < y$, then $0 < x^2 < y^2$*

Proof. Let x and y be real numbers such that $0 < x < y$. Then $x > 0$ and $y > 0$. By O4.1, we obtain

$$\begin{array}{ccccc} 0 \cdot x & < & x \cdot x & < & y \cdot x \\ 0 & < & x^2 & < & xy \end{array}$$

and

$$\begin{array}{ccccc} 0 \cdot y & < & x \cdot y & < & y \cdot y \\ 0 & < & xy & < & y^2. \end{array}$$

Then $0 < x^2 < xy$ and $xy < y^2$. By Transitive Property, $0 < x^2 < y^2$. □

Theorem 1.1.10 *Let a, b, c and d be real numbers.*

$$\textit{If } a < b \textit{ and } c < d, \textit{ then } \quad a + c < b + d.$$

Proof. Let a, b, c and d be real numbers. Assume that $a < b$ and $c < d$. By O3, we get

$$a + c < b + c \quad \text{and} \quad b + c < b + d.$$

By Transitive Property, $a + c < b + d$. □

Theorem 1.1.11 *Let a, b, c and d be real numbers.*

$$\textit{If } 0 < a < b \textit{ and } 0 < c < d, \textit{ then } \quad ac < bd.$$

Proof. Let a, b, c and d be real numbers. Assume that $0 < a < b$ and $0 < c < d$. Then $b > 0$ and $c > 0$. By O4.1, we get

$$ac < bc \quad \text{and} \quad bc < bd.$$

By Transitive Property, $ac < bd$. □

Theorem 1.1.12 *If $a \in \mathbb{R}$, prove that*

$$a \neq 0 \quad \textit{implies} \quad a^2 > 0.$$

In particular, $-1 < 0 < 1$.

Proof. Let a be a real number. Assume that $a \neq 0$. By Trichotomy Property (O1), $a > 0$ or $a < 0$.

Case $a > 0$. By O4.1, $a \cdot a > 0 \cdot a$. So, $a^2 > 0$.

Case $a < 0$. By O4.2, $a \cdot a > 0 \cdot a$. So, $a^2 > 0$.

Moreover, we see that $1 \neq 0$. So, $1 = 1^2 > 0$. By cancellation for addition,

$$1 + (-1) > 0 + (-1).$$

From F7, we obtain $0 > -1$. Thus, $-1 < 0 < 1$. □

Example 1.1.13 *If $x \in \mathbb{R}$, prove that $x > 0$ implies $x^{-1} > 0$.*

Proof. Let $x \in \mathbb{R}$ such that $x > 0$. Then $x^{-1} \neq 0$. By Theorem 1.1.12, $(x^{-1})^2 > 0$. Thus,

$$x^{-1} = x \cdot x^{-2} = x \cdot (x^{-1})^2 > 0 \cdot (x^{-1})^2 = 0.$$

□

Example 1.1.14 *If $x \in \mathbb{R}$, prove that $x < 0$ implies $x^{-1} < 0$.*

Proof. Let $x \in \mathbb{R}$ such that $x < 0$. Then $x^{-1} \neq 0$. By Theorem 1.1.12, $(x^{-1})^2 > 0$. Thus,

$$x^{-1} = x \cdot x^{-2} = x \cdot (x^{-1})^2 < 0 \cdot (x^{-1})^2 = 0.$$

□

Theorem 1.1.15 *Let a and b be real numbers such that $0 < a < b$. Then*

$$\frac{1}{b} < \frac{1}{a}.$$

Proof. Let a and b be real numbers such that $0 < a < b$. Then $ab > 0$. So, $\dfrac{1}{ab} > 0$.

By O4.1, we obtain

$$\begin{aligned} 0 \cdot \frac{1}{ab} &< a \cdot \frac{1}{ab} < b \cdot \frac{1}{ab} \\ 0 &< \frac{1}{b} < \frac{1}{a}. \end{aligned}$$

□

Example 1.1.16 *Let x and y be two distinct real numbers. Prove that*

$$\frac{x+y}{2} \text{ lies between } x \text{ and } y.$$

Proof. Let x and y be two distinct real numbers.

By Trinochomy rule, $x \neq y$. WLOG $x < y$. Then $x + x < x + y$ and $x + y < y + y$.

By transitive rule,

$$\begin{aligned} 2x < x + y &< 2y \\ x < \frac{x+y}{2} &< y. \end{aligned}$$

□

ABSOLUTE VALUE.

Definition 1.1.17 (**Absolute Value**) *The **absolute value** of a number $a \in \mathbb{R}$ is a the number*

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 1.1.18 (**Positive Definite**) *For all $a \in \mathbb{R}$,*

1. $|a| \geq 0$
2. $|a| = 0$ *if and only if* $a = 0$

Proof. Let a be a real number.

1. Case $a = 0$. Then $|a| = |0| = 0 \geq 0$.

   Case $a > 0$. Then $|a| = a > 0$.

   Case $a < 0$. Then $|a| = -a = (-1)a > (-1)0 = 0$.

Hence, $|a| \geq 0$.

2. It's obviously by definition. □

Theorem 1.1.19 (**Multiplicative Law**) *For all $a, b \in \mathbb{R}$,*

$$|ab| = |a||b|.$$

Proof. Let a and b be real numbers.

Case $a = 0$ or $b = 0$. Then $ab = 0$ and $|a| = 0$ or $|b| = 0$. So, $|ab| = |0| = 0 = |a||b|$.

Case $a > 0$ and $b > 0$. Then $ab > 0$, $|a| = a$ and $|b| = b$. So, $|ab| = ab = |a||b|$.

Case $a > 0$ and $b < 0$. Then $ab < 0$, $|a| = a$ and $|b| = -b$. So, $|ab| = -ab = a(-b) = |a||b|$.

Case $a < 0$ and $b > 0$. Then $ab < 0$, $|a| = -a$ and $|b| = b$. So, $|ab| = -ab = (-a)b = |a||b|$.

Case $a < 0$ and $b < 0$. Then $ab > 0$, $|a| = -a$ and $|b| = -b$. So, $|ab| = ab = (-a)(-b) = |a||b|$.

Hence, $|ab| = |a||b|$. □

Theorem 1.1.20 (**Symmetric Law**) *For all $a, b \in \mathbb{R}$,*

$$|a-b| = |b-a|.$$

Moreover, $|a| = |-a|$.

Proof. Let a and b be real numbers. By Multiplicative Law, it implies that

$$|a-b| = |-(-a)+(-b)| = |(-1)(-a)+(-1)b| = |(-1)(-a+b)|$$
$$= |-1||-a+b| = 1 \cdot |-a+b| = |-a+b| = |b-a|.$$

For $b = 0$, we obtain $|a| = |-a|$. □

Example 1.1.21 *Show that* $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ *for all $x \neq 0$.*

Proof. Let x be a non-zero real number.

Case $x > 0$. Then $|x| = x$ and $\frac{1}{x} > 0$. So, $\left|\frac{1}{x}\right| = \frac{1}{x} = \frac{1}{|x|}$.

Case $x < 0$. Then $|x| = -x$ and $\frac{1}{x} < 0$. So, $\left|\frac{1}{x}\right| = -\frac{1}{x} = \frac{1}{-x} = \frac{1}{|x|}$.

□

Theorem 1.1.22 *Let $a, b \in \mathbb{R}$. Show that*

1. $|a^2| = a^2$
2. $a \leq |a|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ *when $b \neq 0$*

Proof. Let $a, b \in \mathbb{R}$. By Theorem 1.1.12, $a^2 \geq 0$. So, $|a^2| = a^2$.

Case $a = 0$. Then $a = 0 \leq 0 = |0| = |a|$.

Case $a > 0$. Then $a \leq a = |a|$.

Case $a < 0$. Then $-a > 0$. So, $a < 0 < -a = |a|$.

Thus, $a \leq |a|$. Use Multiplicative law and Example 1.1.21 to 3, we have

$$\left|\frac{a}{b}\right| = |ab^{-1}| = |a||b^{-1}| = |a| \cdot \left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}$$

□

Theorem 1.1.23 *Let $a \in \mathbb{R}$ and $M \geq 0$. Then*

$$|a| \leq M \quad \textit{if and only if} \quad -M \leq a \leq M$$

Proof. Let $a \in \mathbb{R}$ and $M \geq 0$.

Assume that $|a| \leq M$. By definition, $|a| = \pm a$. Then

$$a \leq M \quad \text{and} \quad -a \leq M.$$

We obtain $a \geq -M$. Thus, $-M \leq a \leq M$.

Conversely, assume that $-M \leq a \leq M$. Then

$$-M \leq a \quad \text{and} \quad a \leq M.$$

So, $M \geq -a$. Thus, $|a| = \pm a \leq M$. □

Corollary 1.1.24 *For all $a \in \mathbb{R}$,* $-|a| \leq a \leq |a|$.

Proof. Choose $M = |a| \geq 0$ in Theorem 1.1.23, we obtain this Corollary. □

INTERVAL.

Let a and b real numbers. A **closed interval** is a set of the form

$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ $\qquad (-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$

$[a, \infty) := \{x \in \mathbb{R} : a \leq x\}$ $\qquad (-\infty, \infty) := \mathbb{R},$

and an **open interval** is a set of the form

$(a, b) := \{x \in \mathbb{R} : a < x < b\}$ $\qquad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$

$(a, \infty) := \{x \in \mathbb{R} : a < x\}$ $\qquad (-\infty, \infty) := \mathbb{R}.$

By an **interval** we mean a closed interval, an open interval, or a set of the form

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\} \quad \text{or} \quad (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Notice, then, that when $a < b$, then intervals $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) correspond to line segments on the real line, but when $b < a$, these interval are all the empty set.

Example 1.1.25 *Solve* $|x-1| \leq 1$ *for* $x \in \mathbb{R}$ *in interval form.*

Solution. By Theorem 1.1.23, $-1 < x - 1 < 1$. So,

$$0 < x < 2.$$

Thus, $x \in (0, 2)$.

Example 1.1.26 *Show that if* $|x| < 1$, *then* $|x^2 + x| < 2$.

Solution. Let $|x| < 1$. Then $-1 < x < 1$. So, $0 < x + 1 < 2$. We obtain

$$-2 < 0 < x + 1 < 1 \quad \longrightarrow \quad |x + 1| < 2.$$

Therefore,

$$|x^2 + x| = |x(x + 1)| = |x||x + 1| < 1 \cdot 2 = 2.$$

Example 1.1.27 *Show that if* $|x - 1| < 2$, *then* $\dfrac{1}{|x|} > 1$.

Solution. Let $|x - 2| < 1$. Then $-1 < x - 2 < 1$. So, $1 < x < 3$. We obtain

$$|x| > 1.$$

Therefore, $\dfrac{1}{|x|} > 1$.

Theorem 1.1.28 (**Triangle Inequality**) *Let* $a, b \in \mathbb{R}$. *Then,*

$$|a + b| \leq |a| + |b|.$$

Proof. Let $a, b \in \mathbb{R}$. By Corollary 1.1.24,

$$\begin{array}{rcccl} -|a| & \leq & a & \leq & |a| \\ -|b| & \leq & b & \leq & |b| \end{array}$$

Then, $-(|a| + |b|) \leq a + b \leq |a| + |b|$. Therefore, $|a + b| \leq |a| + |b|$. □

Theorem 1.1.29 (**Apply Triangle Inequality**) *Let $a, b \in \mathbb{R}$. Then,*

1. $|a-b| \le |a| + |b|$
2. $|a| - |b| \le |a-b|$
3. $|a| - |b| \le |a+b|$
4. $||a| - |b|| \le |a-b|$

Proof. Let $a, b \in \mathbb{R}$.

1. By Triangle Inequality,

$$|a-b| = |a+(-b)| \le |a| + |-b| = |a| + |b|.$$

2. By Triangle Inequality,

$$|a| = |(a-b)+b| \le |a-b| + |b|.$$

Thus, $|a| - |b| \le |a-b|$.

3. By 2,

$$|a| - |b| = |a| - |-b| \le |a-(-b)| = |a+b|.$$

4. By 2, $|a| \le |a-b| + |b|$. By 3,

$$|b| - |a-b| \le |b+(a-b)| = |a|.$$

Then,

$$\begin{array}{rcccl} |b| - |a-b| & \le & |a| & \le & |a-b| + |b| \\ -|a-b| & \le & |a| - |b| & \le & |a-b| \end{array}$$

Thus, $||a| - |b|| \le |a-b|$.

□

Example 1.1.30 *Show that if* $|x-2| < 1$, *then* $|x| < 3$.

Solution. Let $|x-2| < 1$. By 3 in Theorem 1.1.29,

$$|x| - 2 = |x| - |2| < |x-2| < 1.$$

Therefore, $|x| < 1 + 2 = 3$.

Theorem 1.1.31 *Let $x, y \in \mathbb{R}$. Then*

1. $x < y + \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \leq y$
2. $x > y - \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \geq y$

Proof. Let $x, y \in \mathbb{R}$.

1. Assume that $x < y + \varepsilon$ for all $\varepsilon > 0$ and $x > y$. Then $x - y > 0$. By assumption, we get

   $$x < y + (x - y) = x.$$

   It is imposible. So,$x < y + \varepsilon$ for all $\varepsilon > 0$ if and only if $x \leq y$ Conversely, suppose that there is an $\varepsilon > 0$ such that $x \geq y + \varepsilon$. So,

   $$x \geq y + \varepsilon > y + 0 = y$$

   Thus, $x > y$. We conclude that if $x \geq y$, then $x < y + \varepsilon$ for all $\varepsilon > 0$.

2. Excercise.

□

Corollary 1.1.32 *Let $a \in \mathbb{R}$. Then*

$$|a| < \varepsilon \quad \textit{for all } \varepsilon > 0 \quad \textit{if and only if} \quad a = 0$$

Proof. Use Theorem 1.1.31 by $x = |a|$ and $y = 0$. Thus,

$$|a| < 0 + \varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{if and only if} \quad |a| \leq 0.$$

Since $|a| \geq 0$, $|a| = 0$. By positive definite, $a = 0$. The proof is complete. □

Exercises 1.1

1. Let $a, b \in \mathbb{R}$. Prove that

   1.1 $-(a-b) = b-a$

   1.2 $a(b-c) = ab-ac$

   1.3 $(-a)(-b) = ab$

   1.4 $\dfrac{-a}{b} = \dfrac{a}{-b} = -\dfrac{a}{b}$ when $b \neq 0$

2. Let $a, b \in \mathbb{R}$. Prove that

   2.1 If $a+b=a$, then $x=0$.

   2.2 If $ab=b$ and $b \neq 0$, then $a=1$.

   2.3 If $a^{-1}=a$ and $a \neq 0$, then $a=-1$ or $a=1$.

3. Let $a, b, c, d \in \mathbb{R}$. Prove that

   3.1 if $a<b<0$, then $0<b^2<a^2$.

   3.2 if $a<b<0$, then $\dfrac{1}{b}<\dfrac{1}{a}$.

   3.3 if $a \leq b$ and $a \geq b$, then $a=b$.

   3.4 if $0<a<b$, then $\sqrt{a}<\sqrt{b}$.

4. Solve each of the following inequality for $x \in \mathbb{R}$.

   4.1 $|1-2x| \leq 3$

   4.2 $|3-x|<5$

   4.3 $|x^2-x-1|<x^2$

   4.4 $|x^2-x|<2$

5. Prove that if $0<a<1$ and $b=1-\sqrt{1-a}$, then $0<b<a$.

6. Prove that if $a>2$ and $b=1-\sqrt{1-a}$, then $2<b<a$.

7. Prove that $|x| \leq 1$ implies $|x^2-1| \leq 2|x-1|$.

8. Prove that $-1 \leq x \leq 2$ implies $|x^2+x-2| \leq 4|x-1|$.

9. Prove that $|x| \leq 1$ implies $|x^2-x-2| \leq 3|x+1|$.

10. Prove that $0<|x-1| \leq 1$ implies $|x^3+x-2|<8|x-1|$. Is this true if $0 \leq |x-1|<1$?

11. Let $x, y \in \mathbb{R}$. Prove that if $|x+y| = |x-y|$, then $x|y| + y|x| = 0$.

12. Let $x, y \in \mathbb{R}$. Prove that if $|2x+y| = |x+2y|$, then $|xy| = x^2$.

13. Let $a \in \mathbb{R}$. Prove that $\dfrac{a^2+2}{\sqrt{a^2+1}} \geq 2$.

14. Prove that
$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$
for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$

15. Let $x, y \in \mathbb{R}$. Prove that $x > y - \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.

16. Suppose that $x, a, y, b \in \mathbb{R}$, $|x-a| < \varepsilon$ and $|y-b| < \varepsilon$ for some $\varepsilon > 0$. Prove that

    16.1 $|xy - ab| < (|a| + |b|)\varepsilon + \varepsilon^2$

    16.2 $|x^2y - a^2b| < \varepsilon(|a|^2 + 2|ab|) + \varepsilon^2(|b| + 2|a|) + \varepsilon^2$

17. The **positive part** of an $a \in \mathbb{R}$ is defined by
$$a^+ := \frac{|a| + a}{2}$$
and the **negative part** by
$$a^- := \frac{|a| - a}{2}.$$

    17.1 Prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

    17.2 Prove that $a^+ := \begin{cases} a & : a \geq 0 \\ 0 & : a \leq 0 \end{cases}$ and $a^- := \begin{cases} 0 & : a \geq 0 \\ -a & : a \leq 0 \end{cases}$.

18. Let $a, b \in \mathbb{R}$. The **arithmetic mean** of a, b is $A(a,b) := \dfrac{a+b}{2}$,
the **geometric mean** of $a, b \in (0, \infty)$ is $G(a,b) := \sqrt{ab}$,
and **harmonic mean** of $a, b \in (0, \infty)$ is $H(a,b) := \dfrac{2}{a^{-1} + b^{-1}}$.
Show that

    18.1 if $a, b \in (0, \infty)$. Then $H(a,b) \leq G(a,b) \leq A(a,b)$.

    18.2 if $0 < a \leq b$. Then $a \leq G(a,b) \leq A(a,b) \leq b$.

    18.3 if $0 < a \leq b$. Then, $G(a,b) = A(a,b)$ if and only if $a = b$.

1.2 Well-Ordering Principle

Definition 1.2.1 *A number m is a **least element** of a set $S \subset \mathbb{R}$ if and only if*

$$m \in S \text{ and } m \leq s \text{ for all } s \in S.$$

WELL-ORDERING PRINCIPLE (WOP).

Every nonempty subset of $\mathbb{N}$ has a least element.

$$S \subseteq \mathbb{N} \wedge S \neq \varnothing \rightarrow \exists m \in S \, \forall s \in S, \ m \leq s.$$

Theorem 1.2.2 (**Mathematical Induction**) *Suppose for each $n \in \mathbb{N}$ that $P(n)$ is a statement that satisfies the following two properties:*

(1) Basic step : $P(1)$ is true

(2) Inductive step : For every $k \in \mathbb{N}$ for which $P(k)$ is true, $P(k+1)$ is also true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. We will prove by contradiction. Assume that (1) and (2) are ture and there is an $n_0 \in \mathbb{N}$ such that $P(n_0)$ is false. Define

$$S = \{n \in \mathbb{N} : P(n) \text{ is false } \}.$$

Then, $n_0 \in S \subseteq \mathbb{N}$. By WOP, S has a least element, said $m \in S$.
Since (1) is true, $m \neq 1$. Then $m > 1$ or $m - 1 > 0$. So, $m - 1 \in \mathbb{N}$.
But $m - 1 < m$ and m is the least element in S, so $m - 1 \notin S$. Set

$$k = m - 1 \in \mathbb{N}. \text{ We obtain } P(k) \text{ is true.}$$

By (2), $P(k+1) = P(m)$ is true. This contradicts $m \in S$. □

Example 1.2.3 (**Gauss' formula**) *Prove that*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Proof. For $n = 1$, we get $\sum_{k=1}^{1} k = 1 = \frac{2}{2} = \frac{1(1+1)}{2}$. So, (1) is true.
Assume that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then,

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left[\frac{n}{2} + 1\right] = \frac{(n+1)(n+2)}{2}.$$

So, (2) is true. By Mathematical Induction, Gauss' formula is proved. □

Example 1.2.4 *Prove that* $2^n > n$ *for all* $n \in \mathbb{N}$.

Proof. We will prove by induction on n. For $n = 1$, it is clear $2^1 > 1$. Assume that $2^n > n$ for some $n \in \mathbb{N}$. By inductive hypothesis and the fact that $n \geq 1$,

$$2^{n+1} = 2^n \cdot 2 > 2n = n + n \geq n + 1.$$

So, $2^n > n$ is true for $n + 1$. We conclude by induction that $2^n > n$ holds for $n \in \mathbb{N}$. □

BINOMIAL FORMULA.

Definition 1.2.5 *The notation* $0! = 1$ *and* $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ *for* $n \in \mathbb{N}$ *(called* ***factorial****), define the* ***binomial coefficient n over k*** *by*

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}$$

for $0 \leq k \leq n$ and $n = 0, 1, 2, 3, \ldots$

Theorem 1.2.6 *If* $n, k \in \mathbb{N}$ *and* $1 \leq k \leq n$, *then*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof. Let $n, k \in \mathbb{N}$ and $1 \le k \le n$. We obtain

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{n!k}{(n-k+1)!(k-1)!k} + \frac{n!(n-k+1)}{(n-k+1)(n-k)!k!} \\ &= \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!} = \frac{n!k + n!(n-k+1)}{(n-k+1)!k!} \\ &= \frac{n![k+(n-k+1)]}{(n-k+1)!k!} = \frac{n!(n+1)}{(n-k+1)!k!} = \frac{(n+1)!}{(n-k+1)!k!} = \binom{n+1}{k}.\end{aligned}$$

□

Theorem 1.2.7 (**Binomial formula**) *If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. We will prove by induction on n. The formula is obvious for $n = 1$. Assume that the formula is true for some $n \in \mathbb{N}$. By inductive hypothesis,

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\ &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} \\ &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1} b^k + b^{n+1} \\ &= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] a^{n-k+1} b^k + b^{n+1}\end{aligned}$$

Thus, it follows from Theorem 1.2.6 that

$$(a+b)^{n+1} = a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n-k-1} b^k + b^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

i.e., the formula is true for $n+1$. We conclude by induction that the formula holds for $n \in \mathbb{N}$. □

Exercises 1.2

1. Prove that the following formulas hold for all $n \in \mathbb{N}$.

   1.1 $\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3+6n^2+n$

   1.2 $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

   1.3 $\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$

   1.4 $\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}, \quad a \neq 0$

2. Use the Binomial Formula to prove each of the following.

   2.1 $2^n = \sum_{k=1}^{n} \binom{n}{k}$ for all $n \in \mathbb{N}$.

   2.2 $(a+b)^n \geq a^n + aa^{n-1}b$ for all $n \in \mathbb{N}$ and $a, b \geq 0$.

   2.3 $\left(1+\frac{1}{n}\right)^n \geq 2$ for all $n \in \mathbb{N}$.

3. Let $n \in \mathbb{N}$. Write

$$\frac{(x+h)^n - x^n}{h}$$

   as a sum none of whose terms has an h in the dennominator.

4. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_{n+1} < x_n < 1$ holds for all $n \in \mathbb{N}$.

5. Suppose that $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $2 \leq x_{n+1} \leq x_n \leq x_1$ holds for all $n \in \mathbb{N}$.

6. Suppose that $0 < x_1 < 2$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_n < x_{n+1} < 2$ holds for all $n \in \mathbb{N}$.

7. Prove that each of the following inequalities hold for all $n \in \mathbb{N}$.

   7.1 $n < 3^n$

   7.2 $n^2 \leq 2^n + 1$

   7.3 $n^3 \leq 3^n$

8. Let $0 < |a| < 1$. Prove that $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$.

9. Prove that $0 \leq a < b$ implies $a^n < b^n$ for all $n \in \mathbb{N}$.

1.3 Completeness Axiom

SUPREMUM.

Definition 1.3.1 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be* ***bounded above*** *if and only if*

$$\text{there is an } M \in \mathbb{R} \text{ such that } a \leq M \text{ for all } a \in A$$

2. *A number M is called an* ***upper bound*** *of the set A if and only if*

$$a \leq M \text{ for all } a \in A$$

3. *A number s is called a* ***supremum*** *of the set A if and only if*

$$s \text{ is an upper bound of } A \text{ and } s \leq M \text{ for all upper bound } M \text{ of } A$$

In this case we shall say that A has a supremum s and shall write $s = \sup A$

Example 1.3.2 *Fill the blanks of the following table.*

Sets	Bounded above	Set of Upper bound	Supremum
$A = [0, 1]$	Yes	$[1, \infty)$	1
$A = (0, 1)$	Yes	$[1, \infty)$	1
$A = \{1\}$	Yes	$[1, \infty)$	1
$A = (0, \infty)$	No	$\varnothing$	None
$A = (-\infty, 0)$	Yes	$[0, \infty)$	0
$A = \mathbb{N}$	No	$\varnothing$	None
$A = \mathbb{Z}$	No	$\varnothing$	None

Example 1.3.3 *Show that* $\sup A = 1$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Solution.

1. For $A = [0, 1]$. Since $a \leq 1$ for all $a \in A$, 1 is an upper bound of A.

   Let M be an upper bound of A. Then,

   $$a \leq M \quad \text{for all } a \in A$$

   Since $1 \in A$, $1 \leq M$. Thus, $\sup A = 1$.

2. For $A = (0, 1)$. Since $a < 1 \leq 1$ for all $a \in A$, 1 is an upper bound of A.

   Suppose that there is an upper bound M_0 of A such that $M_0 < 1$. Then,

   $$a < M_0 \quad \text{for all } a \in A$$

   But $0 < a < M_0 < \dfrac{M_0 + 1}{2} < 1$, so $\dfrac{M_0 + 1}{2}$ belongs to A. It is imposible because M_0 is an upper bound of A. Hence, there is no upper bound of A such that it is less that 1. We conclude that $\sup A = 1$.

Theorem 1.3.4 *If a set has one upper bound, then it has infinitely many upper bounds.*

Proof. Let M_0 be an upper bound of a set A. We set

$$M := M_0 + k \quad \text{for all } k \in \mathbb{N}.$$

Then, $M > M_0$ for all $k \in \mathbb{N}$. So, M is another upper bound of A depending on k.

This reason shows that it has infinitely many upper of A. $\square$

Theorem 1.3.5 *If a set has a supremum, then it has only one supremum.*

Proof. Let s_1 and s_2 be suprema of the same of a set A. Then, s_1 and s_2 are upper bounds of A. By definition of supremum, we obtain

$$s_1 \leq s_2 \quad \text{and} \quad s_2 \leq s_1.$$

Therefore, $s_1 = s_2$. □

Theorem 1.3.6 (**Approximation Property for Supremum (APS)**) *If A has a supremum and $\varepsilon > 0$ is any positive number, then there is a point $a \in A$ such that*

$$\sup A - \varepsilon < a \leq \sup A$$

Proof. We will prove by contradiction. Assume that A has an infimum, say s. Suppose that there a positve $\varepsilon_0 > 0$ such that

$$a \leq s - \varepsilon_0 \quad \text{or} \quad a > s \qquad \text{for all } a \in A$$

In this case $a > s$, it is imposible beacause s is an upper bound of A.
From $a \leq s - \varepsilon_0$ for all $a \in A$, it means that $s - \varepsilon_0$ is an upper bound of A. But

$$s - \varepsilon_0 < s$$

It's imposible because s is the least upper bound of A. □

Theorem 1.3.7 *If $A \subset \mathbb{N}$ has a supremum, then* $\sup A \in A$.

Proof. Assume that $A \subset \mathbb{N}$ has a supremum, say s. Apply APS to choose an $x_0 \in A$ such that

$$s - 1 < x_0 \leq s.$$

If $x_0 = s$, then $s \in A$. In this case $s - 1 < x_0 < s$. Apply again APS to choose $x_1 \in A$ such that

$$\begin{array}{ccccc} x_0 & < & x_1 & < & s \\ 0 & < & x_1 - x_0 & < & s - x_0. \end{array}$$

Since $x_0, x_1 \in \mathbb{N}$ and $x_0 \neq x_1$, $x_1 - x_0 \geq 1$. From $s - 1 < x_0$ and $x_1 < s$, we get

$$(s-1) + x_1 < x_0 + s.$$

So, $x_1 - x_0 < 1$. It contradicts to $x_1 - x_0 \geq 1$. Thus, this case is false. □

COMPLETENESS AXIOM.

If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then A has a supremum.

Theorem 1.3.8 *The set of natural numbers is not bounded above.*

Proof. Suppose that $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is not a nonempty set by Completeness Axiom, $\mathbb{N}$ has a supremum, say s. Then

$$n \leq s \quad \text{for all } n \in \mathbb{N}.$$

If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. So, $n + 1 \leq s \quad$ for all $n \in \mathbb{N}$, i.e.,

$$n \leq s - 1 \quad \text{for all } n \in \mathbb{N}.$$

Thus, $s - 1$ is an upper bound of $\mathbb{N}$. We obatain $s \leq s - 1$ or $0 < -1$. It is imposible. □

Theorem 1.3.9 (**Archimedean Properties (AP)**) *For each $x \in \mathbb{R}$, the following statements are true.*

1. *There is an integer $n \in \mathbb{N}$ such that $x < n$.*
2. *If $x > 0$, there there is an integer $n \in \mathbb{N}$ such that $\dfrac{1}{n} < x$.*

Proof. Suppose that there is an $x \in \mathbb{R}$ such that $x \geq n$ for all $n \in \mathbb{N}$. It means that x is an upper bound of $\mathbb{N}$. This is contradiction Theorem 1.3.8. Thus, part 1 is proved.
Next, we assume that $x > 0$. Then $\dfrac{1}{x} \in \mathbb{R}$. By 1, there is an $n \in \mathbb{N}$ such that $\dfrac{1}{x} < n$. Thus,

$$\frac{1}{n} < x.$$

The proof of Archimedean Properties is complete. □

Theorem 1.3.10 *Let $x \in \mathbb{R}$. Then*

$$|x| < \frac{1}{n} \quad \textit{for all } n \in \mathbb{N} \quad \textit{if and only if} \quad x = 0$$

Proof. Let $x \in \mathbb{R}$. Assume that $|x| < \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By AP, there an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$. By assumption, we obtain

$$|x| < \frac{1}{N} < \varepsilon.$$

From Corollary 1.1.32 , it implies that $x = 0$. Conversely, it is obvious. □

Example 1.3.11 *Let* $A = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

Proof. For each $n \in \mathbb{N}$, we get $n \geq 1$. So, $\dfrac{1}{n} \leq 1$. Thus, 1 is an upper bound of A.
Let M be any upper bound of A. Then

$$a \leq M \quad \text{for all } a \in A.$$

For $n = 1$, we have $1 = \frac{1}{1} \in A$. So, $1 \leq M$. Hence, $\sup A = 1$. □

Example 1.3.12 *Let* $A = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

Proof. Since $0 < n < n + 1$ for all $n \in \mathbb{N}$, $\dfrac{n}{n+1} < 1 \quad$ for all $n \in \mathbb{N}$.
Thus, 1 is an upper bound of A.
Suppose that that there is an upper bound u_0 of A such that $u_0 < 1$.
Since $u_0 < 1$, $1 - u_0 > 0$. By AP, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < 1 - u_0.$$

Since $n_0 + 1 > n_0 > 0$, $\dfrac{1}{n_0 + 1} < \dfrac{1}{n_0}$. We obtain

$$\frac{1}{n_0 + 1} < 1 - u_0$$
$$u_0 < 1 - \frac{1}{n_0 + 1} = \frac{n_0}{n_0 + 1}$$

So, u_0 is not upper bound of A. This is contradiction. Therefore, $\sup A = 1$. □

Theorem 1.3.13 *If $x \in \mathbb{R}$, then there is an $n \in \mathbb{Z}$ such that*

$$n - 1 \leq x < n.$$

Proof. Let $x \in \mathbb{R}$. If $x = 0$, we choose $n = 1$. We are done.

Case 1. $x > 0$. Define $S = \{n \in \mathbb{N} : n > x\} \subseteq \mathbb{N}$. By AP, $S \neq \varnothing$. From WOP, S has the least element, say n_0. Since $n_0 - 1 < n_0$, $n_0 - 1 \notin A$. So, $n_0 - 1 \leq x$. Thus,

$$n_0 - 1 \leq x < n_0.$$

The proof is complete in this case.

Case 2. $x < 0$. Then $-x > 0$. By Case 1, there is an $m \in \mathbb{N}$ such that $m - 1 \leq -x < m$. Then

$$-m < x \leq -m + 1.$$

If $x = -m + 1$, we choose $n = -m + 2$. So,

$$n - 1 = -m + 1 = x < n \text{ or } n - 1 \leq x < n.$$

If $-m < x < -m+1$, we choose $n = -m+1$. So, $n-1 < x < n$. It implies that $n-1 \leq x < n$. □

Theorem 1.3.14 (**Density of Rationals**) *If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is a rational number r such that*

$$a < r < b.$$

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Then $b - a > 0$.

By AP, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{n} < b - a$. It follows that

$$na + 1 < nb.$$

By Theorem 1.3.13, there is an $m \in \mathbb{Z}$ such that $m - 1 \leq na < m$. It implies that

$$na < m \leq na + 1 < nb.$$

Set $r := \dfrac{m}{n}$. We obtain $a < r < b$. □

Theorem 1.3.15 $\sqrt{2}$ *is irrational.*

Proof. Assume that $\sqrt{2}$ is a rational number. Then there are two integers p and q such that

$$\sqrt{2} = \frac{p}{q} \text{ when } q \neq 0 \text{ and } \gcd(p, q) = 1.$$

We have $2q^2 = p^2$. It implies that p is an even number. Then there is an $k \in \mathbb{Z}$ such that $p = 2k$. So,

$$2q^2 = (2k)^2 = 4k^2$$
$$q^2 = 2k^2$$

It implies again that q is an even number. Thus, $\gcd(p, q) \neq 1$. This is contradiction. □

Theorem 1.3.16 (**Density of Irrationals**) *If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is an irrational number t such that*

$$a < t < b.$$

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. By the Density of Rational, there is an $r \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. It follows that

$$a < r\sqrt{2} < b.$$

If $r \neq 0$, then $t := r\sqrt{2}$ is irrational (see Exercise). It is done.
Case $r = 0$. By the Density of Rational, there is an $s \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < 0 < s < \frac{b}{\sqrt{2}}$. It follows that

$$a < s\sqrt{2} < b.$$

Set $t = s\sqrt{2}$, irrational. Thus, the proof is complete. □

INFIMUM.

Definition 1.3.17 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be **bounded below** if and only if*

$$\text{there is an } m \in \mathbb{R} \text{ such that } m \leq a \text{ for all } a \in A$$

2. *A number m is called a **lower bound** of the set A if and only if*

$$m \leq a \quad \text{for all } a \in A$$

3. *A number ℓ is called an **infimum** of the set A if and only if*

$$\ell \text{ is a lower bound of } A \text{ and } m \leq \ell \text{ for all lower bound } m \text{ of } A$$

In this case we shall say that A has an infimum s and shall write $\ell = \inf A$

4. *A is said to be **bounded** if and only if it is bounded above and below.*

Example 1.3.18 *Fill the blanks of the following table.*

Sets	Bounded below	Set of Lower bound	Infimum	Bounded
$A = [0, 1]$	Yes	$(-\infty, 0]$	0	Yes
$A = (0, 1)$	Yes	$(-\infty, 0]$	0	Yes
$A = \{1\}$	Yes	$(-\infty, 1]$	1	Yes
$A = (0, \infty)$	Yes	$(-\infty, 0]$	0	No
$A = (-\infty, 0)$	No	$\varnothing$	None	No
$A = \mathbb{N}$	Yes	$(-\infty, 1]$	1	No
$A = \mathbb{Z}$	No	$\varnothing$	None	No

Example 1.3.19 *Show that* $\inf A = 0$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Solution.

1. For $A = [0, 1]$. Since $a \geq 0$ for all $a \in A$, 1 is a lower bound of A.

   Let m be a lower bound of A. Then,

   $$m \leq a \quad \text{for all } a \in A$$

   Since $0 \in A$, $0 \leq M$. Thus, $\inf A = 0$.

2. For $A = (0, 1)$. Since $a > 0 \geq 0$ for all $a \in A$, 0 is a lower bound of A.

   Suppose that there is a lower bound m_0 of A such that $m_0 > 0$. Then,

   $$m_0 \leq a \quad \text{for all } a \in A$$

   But $0 < \dfrac{m_0}{2} < m_0 \leq a$, so $\dfrac{m_0}{2}$ belongs to A. It is imposible because m_0 is a lower bound of A. Hence, there is no lower bound of A such that it is greater that 0. We conclude that $\inf A = 0$.

Example 1.3.20 *Let* $A = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = 0$.

Proof. For each $n \in \mathbb{N}$, we get $n > 0$. So, $\dfrac{1}{n} > 0$. Thus, 0 is a lower bound of A.
Suppose that that there is a lower bound m_0 of A such that $m_0 > 0$.
By AP, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < m_0.$$

So, m_0 is not lower bound of A. This is contradiction. Therefore, $\inf A = 0$. □

Example 1.3.21 *Let* $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = \frac{1}{2}$.

Proof. Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $\frac{1}{n} \leq 1$ or $1 + \frac{1}{n} \leq 2$. We obtain

$$\frac{1}{2} \leq \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1}.$$

Thus, $\frac{1}{2}$ is a lower bound of A.

Let m_0 be any lower bound of A. Then

$$m_0 \leq a \quad \text{for all } a \in A.$$

For $n = 1$, we have that $\frac{1}{2} = \frac{1}{1+1}$ belongs to A.

$$m_0 \leq \frac{1}{2}.$$

Therefore, $\inf A = \frac{1}{2}$. □

Theorem 1.3.22 (**Approximation Property for Infimum (API)**) *If A has an infimum and $\varepsilon > 0$ is any positive number, then there is a point $a \in A$ such that*

$$\inf A \leq a < \inf A + \varepsilon.$$

Proof. Assume that A has an infimum, say ℓ_0. Suppose that there a positve $\varepsilon_0 > 0$ such that

$$a < \ell_0 \quad \text{or} \quad a \geq \ell_0 + \varepsilon_0 \text{ for all } a \in A$$

In this case $a < \ell_0$, it is imposible beacause ℓ_0 is a lower bound of A.

From $a \geq \ell_0 + \varepsilon_0$ for all $a \in A$, it means that $\ell_0 + \varepsilon_0$ is a lower bound of A. But

$$\ell_0 + \varepsilon_0 > \ell_0$$

It's imposible because ℓ_0 is the greatest lower bound of A. □

Exercises 1.3

1. Find the infimum and supremum of each the following sets.

   1.1 $A = [0, 2)$

   1.2 $A = \{4, 3, 1, 5\}$

   1.3 $A = \{x \in \mathbb{R} : |x - 1| < 2\}$

   1.4 $A = \{x \in \mathbb{R} : |x + 1| < 1\}$

   1.5 $A = \{1 + (-1)^n : n \in \mathbb{N}\}$

   1.6 $A = \left\{\dfrac{1}{n} - (-1)^n : n \in \mathbb{N}\right\}$

   1.7 $A = \left\{1 + \dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

   1.8 $A = \left\{\dfrac{n+1}{n} : n \in \mathbb{N}\right\}$

   1.9 $A = \left\{\dfrac{n^2+n}{n^2+1} : n \in \mathbb{N}\right\}$

   1.10 $A = \left\{\dfrac{n(-1)^n + 1}{n+2} : n \in \mathbb{N}\right\}$

2. Find $\inf A$ and $\sup A$ with proving them.

   2.1 $A = [-1, 1]$

   2.2 $A = (-1, 2]$

   2.3 $A = (-1, 0) \cup (1, 2)$

   2.4 $A = \{1, 2, 3\}$

   2.5 $A = \left\{\dfrac{n}{n+2} : n \in \mathbb{N}\right\}$

   2.6 $A = \left\{\dfrac{n-2}{n+2} : n \in \mathbb{N}\right\}$

   2.7 $A = \left\{\dfrac{n}{n^2+1} : n \in \mathbb{N}\right\}$

   2.8 $A = \{(-1)^n : n \in \mathbb{N}\}$

3. Let $A = \left\{1 - \dfrac{n}{n^2+2} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

4. Let $A = \left\{2 - \dfrac{n}{n^2+1} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

5. If a set has one lower bound, then it has infinitely many lower bounds.

6. Prove that if A is a nonempty bounded subset of $\mathbb{Z}$, then both $\sup A$ and $\inf A$ exist and belong to A.

7. Prove that for each $a \in \mathbb{R}$ and each $n \in \mathbb{N}$ there exists a rational r_n such that

$$|a - r_n| < \frac{1}{n}.$$

8. Let r be a rational number and s be an irrational number. Prove that

   8.1 $r+s$ is an irrational number.

   8.2 if $r \neq 0$, then rs is always an irrational number.

9. Let $\sqrt{K} \in \mathbb{Q}^c$ and $a, b, x, y \in \mathbb{Z}$. Prove that

$$\text{if } a+b\sqrt{K} = x+y\sqrt{K}, \text{ then } a=x \text{ and } b=y.$$

10. Show that a lower bound of a set need not be unique but the infimum of a given set A is unique.

11. Show that if A is a nonoempty subset of $\mathbb{R}$ that is bounded below, then A has a finite infimum.

12. Prove that if x is an upper bound of a set $A \subseteq \mathbb{R}$ and $x \in A$, then x is the supremum of A.

13. Suppose $E, A, B \subset \mathbb{R}$ and $E = A \cup B$. Prove that if E has a supremum and both A and B are nonempty, then Sup$A$ and $\sup B$ both exist, and $\sup E$ is one of the numbers Sup$A$ or $\sup B$.

14. (**Monotone Property**) Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$. Prove that

    14.1 if B has a supremum, then $\sup A \leq \sup B$

    14.2 if B has an infimum, then $\inf B \leq \inf A$

15. Define the **reflection** of a set $A \subseteq \mathbb{R}$ by

$$-A := \{-x : x \in A\}$$

    Let $A \subseteq \mathbb{R}$ be nonempty. Prove that

    15.1 A has a supremum if and only if $-A$ has and infimum, in which case

$$\inf(-A) = -\sup A.$$

    15.2 A has an infimum if and only if $-A$ has and supremum, in which case

$$\sup(-A) = -\inf A.$$

1.4 Functions and Inverse functions

Review notation $f : X \to Y$ that means a fuction form X to Y, each $x \in X$ is assigned a unique $y = f(x) \in Y$, there is nothing that keeps two x's from being assigned to the same y, and nothing that say every $y \in Y$ corresponds to some $x \in X$, i.e., f is a fuction if and only if for each $(x_1, y_1), (x_2, y_2)$ belong to f,

$$\text{if } x_1 = x_2 \text{ , then } y_2 = y_2.$$

Definition 1.4.1 *Let f be a function from a set X into a set Y.*

1. *f is said to be* ***one-to-one (1-1)*** *on X if and only if*

$$x_1, x_2 \in X \textit{ and } f(x_1) = f(x_2) \textit{ imply } x_1 = x_2.$$

2. *f is said to take X* ***onto*** *Y if and only if*

$$\textit{for each } y \in Y \textit{ there is an } x \in X \textit{ such that } y = f(x).$$

Example 1.4.2 *Show that $f(x) = 2x + 1$ is 1-1 from $\mathbb{R}$ onto $\mathbb{R}$.*

Solution. Let x_1 and x_2 be reals such that $f(x_1) = f(x_2)$. Then,

$$2x_1 + 1 = 2x_2 + 1$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

So, f is 1-1. Let $y \in \mathbb{R}$. Choose $x = \dfrac{y-1}{2} \in \mathbb{R}$. Then,

$$f(x) = 2x + 1 = 2\left(\frac{y-1}{2}\right) + 1 = y$$

Thus, f takes $\mathbb{R}$ onto $\mathbb{R}$.

Theorem 1.4.3 *Let X and Y be sets and $f : X \to Y$. Then f is 1-1 from X onto Y if and only if there is a unique function g from Y onto X that satisfies*

1. $f(g(y)) = y, \quad y \in Y$

   and

2. $g(f(x)) = x, \quad x \in X$

Proof. Suppose that f is 1-1 and onto. For each $y \in Y$ choose the unique $x \in X$ such that $f(x) = y$, and define

$$g(y) := x.$$

It is clear that g take Y onto X. By construction, 1 and 2 are satisfied.

Conversely, suppose that there a function g from Y onto X that satisfies 1 and 2.

Let $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Then it follows from 2 that

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

Thus f is 1-1 on X. Let $y \in Y$ and choose $x = g(y)$. Then 1 implies that

$$f(x) = f(g(y)) = y.$$

Thus f takes X onto Y.

Finally, suppose that h is another function that satisfies 1 and 2, and $y \in Y$. Choose $x \in X$ such that $f(x) = y$. Then, by 2,

$$h(y) = h(f(x)) = x = g(f(x)) = g(y);$$

i.e., $h = g$ on Y. It follows that the function is unique. □

If f is 1-1 from a set X onto a set Y, we shall say that f has an **inverse function**. We shall call the function g given in Theorem 1.4.3 the **inverse** of f, and denote it by f^{-1}. Then

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Example 1.4.4 *Find inverse function of $f(x) = 2x + 1$.*

Solution. By Example 1.4.2, f is 1-1 from $\mathbb{R}$ onto $\mathbb{R}$. Then,

$$f^{-1}(2x+1) = f^{-1}(f(x)) = x$$

Substitue $x := \frac{x-1}{2}$. We obtain

$$f^{-1}(x) = f^{-1}\left(2 \cdot \frac{x-1}{2} + 1\right) = \frac{x-1}{2}.$$

Example 1.4.5 *Let $f(x) = e^x - e^{-x}$.*

1. *Show that f is 1-1 from $\mathbb{R}$ onto $\mathbb{R}$.*

2. *Find a formula of $f^{-1}(x)$.*

Solution. Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. WLOG $x_1 > x_2$. Then $e^{x_1} > e^{x_2}$. Since $-x_1 < -x_2$, $e^{-x_1} < e^{-x_2}$. We obtain

$$e^{x_2} + e^{-x_1} > e^{x_1} + e^{-x_2}$$

$$f(x_2) = e^{x_2} - e^{-x_2} > e^{x_1} - e^{-x_1} = f(x_1)$$

Then $f(x_1) \neq f(x_2)$. Thus f is 1-1 on $\mathbb{R}$. Let $y \in \mathbb{R}$. Choose $x = \ln\left(\frac{y+\sqrt{y^2+4}}{2}\right)$. Then

$$f(x) = e^{\ln\left(\frac{y+\sqrt{y^2+4}}{2}\right)} - e^{-\ln\left(\frac{y+\sqrt{y^2+4}}{2}\right)} = \frac{y+\sqrt{y^2+4}}{2} - \frac{2}{y+\sqrt{y^2+4}} = y.$$

Thus, f takes $\mathbb{R}$ onto $\mathbb{R}$. Consider

$$f^{-1}(e^x - e^{-x}) = f^{-1}(f(x)) = x.$$

Substitue $x := \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)$. We obtain

$$\begin{aligned} f^{-1}\left(e^{\ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)} - e^{-\ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)}\right) &= \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right) \\ f^{-1}\left(\frac{x+\sqrt{x^2+4}}{2} - \frac{2}{x+\sqrt{x^2+4}}\right) &= \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right) \\ f^{-1}(x) &= \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right). \end{aligned}$$

Exercises 1.4

1. For each of the following, prove f is 1-1 from A onto A. Find a formula for f^{-1}.

   1.1 $f(x) = 3x - 7$: $A = \mathbb{R}$

   1.2 $f(x) = x^2 - 2x - 1$: $A = (1, \infty)$

   1.3 $f(x) = 3x - |x| + |x - 2|$: $A = \mathbb{R}$

   1.4 $f(x) = x|x|$: $A = \mathbb{R}$

   1.5 $f(x) = e^{\frac{1}{x}}$: $A = (0, \infty)$

   1.6 $f(x) = \tan x$: $A = (-\frac{\pi}{2}, \frac{\pi}{2})$

   1.7 $f(x) = \dfrac{x}{x^2 + 1}$: $A = [-1, 1]$

2. Let $f(x) = x^2 e^{x^2}$ where $x \in \mathbb{R}$. Show that f is 1-1 on $(0, \infty)$.

3. Suppose that A is finite and f is 1-1 from A onto B. Prove that B is finite.

4. Prove that there a fuction f that is 1-1 from $\{2, 4, 6, ...\}$ onto $\mathbb{N}$.

5. Prove that there a fuction f that is 1-1 from $\{1, 3, 5, ...\}$ onto $\mathbb{N}$.

6. Suppose that $n \in \mathbb{N}$ and $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$.

   6.1 Prove that ϕ is 1-1 if and only if ϕ in onto.

   6.2 Suppose that A is finite and $f : A \to A$. Prove that

   $$f \text{ is 1-1 on } A \text{ if and only if } f \text{ takes } A \text{ onto } A.$$

7. Let $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a 1-1 function. Show that $\displaystyle\sum_{x=1}^{n} f(x) = n!$.

Chapter 2

Sequences in $\mathbb{R}$

2.1 Limits of sequences

An **infinite sequence** (more briefly, a sequence) is a function whose domain in $\mathbb{N}$. A sequence f whose term are $x_n := f(n)$ will be defined by

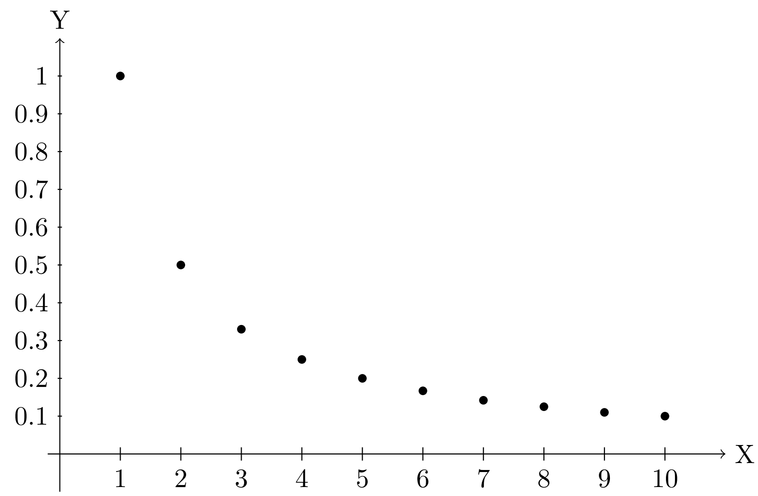
$$x_1, x_2, x_3, ... \quad \text{or} \quad \{x_n\}_{n\in\mathbb{N}} \quad \text{or} \quad \{x_n\}_{n=1}^{\infty} \quad \text{or} \quad \{x_n\}.$$

Example 2.1.1 *Use notation to represents the following sequences.*

1. $1, 2, 3, ...$ represents the sequence $\{n\}_{n\in\mathbb{N}}$

2. $1, -1, 1, -1, ...$ represents the sequence $\{(-1)^n\}$

Example 2.1.2 *Sketch graph of* $\{x_n\}$ *and guess* x_n *if* n *go to infinity where* $x_n = \dfrac{1}{n}$



By the graph, we will see that x_n approaches to ZERO as n go to infinity.

Definition 2.1.3 *A sequence of real numbers $\{x_n\}$ is said to* ***converge*** *to a real number $a \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad |x_n - a| < \varepsilon.$$

We shall use the following phrases and notations interchangeably:

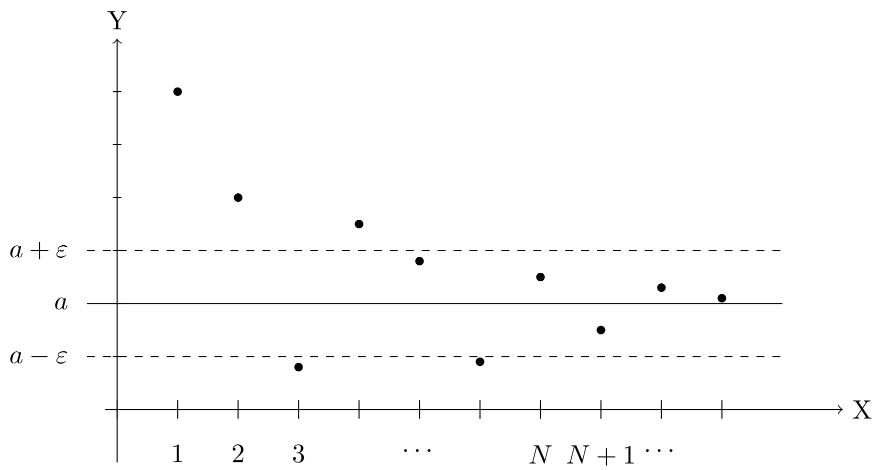
(a) $\{x_n\}$ converges to a;

(b) x_n converges to a;

(c) $\lim_{n\to\infty} x_n = a$;

(d) $x_n \to a$ as $n \to \infty$;

(e) the limit of $\{x_n\}$ exists and equals a.



Theorem 2.1.4 $\lim_{n\to\infty} k = k$ *where k is a constant.*

Proof. Let k be a constant and $\varepsilon > 0$. We can choose whatever $N \in \mathbb{N}$ such that for each $n \geq N$, we always obtain

$$|k - k| = 0 < \varepsilon.$$

So, $\lim_{n\to\infty} k = k$. ☐

Example 2.1.5 *Prove that* $\dfrac{1}{n} \to 0$ *as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Then $\dfrac{1}{n} \leq \dfrac{1}{N}$. We obtain

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\dfrac{1}{n} \to 0$ as $n \to \infty$. ☐

Example 2.1.6 *Prove that* $\lim_{n\to\infty} \frac{n}{n+1} = 1$

Proof. Let $\varepsilon > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n+1 > n \geq N$. So, $\frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N}$. We obtain

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\frac{n}{n+1} \to 1$ as $n \to \infty$. □

Example 2.1.7 *Prove that* $\frac{1}{2^n} \to 0$ *as* $n \to \infty$

Proof. Let $\varepsilon > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Let $n \in \mathbb{N}$ such that $n \geq N$. By Example 1.2.4, $2^n > n$. So, $\frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N}$. We obtain

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\frac{1}{2^n} \to 0$ as $n \to \infty$. □

Example 2.1.8 *Prove that* $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Proof. Let $\varepsilon > 0$. Then $\sqrt{\varepsilon} > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \sqrt{\varepsilon}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Since $n \geq N > 0$, $n^2 \geq N^2$. Then $\frac{1}{n^2} \leq \frac{1}{N^2}$. We obtain

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

Thus, $\frac{1}{n^2} \to 0$ as $n \to \infty$. □

Example 2.1.9 *Prove that* $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

Proof. Let $\varepsilon > 0$. Then $\varepsilon^2 > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon^2$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Since $n \geq N > 0$, $\sqrt{n} \geq \sqrt{N}$. Then $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}$.
Since $\sqrt{n+1} > 0$, $\sqrt{n+1} + \sqrt{n} > \sqrt{n}$. Then $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$. We obtain

$$\begin{aligned}\left|\sqrt{n+1} - \sqrt{n} - 0\right| &= (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.\end{aligned}$$

Thus, $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$. □

Example 2.1.10 *If* $x_n \to 1$ *as* $n \to \infty$*. Prove that*

$$2x_n + 1 \to 3 \text{ as } n \to \infty.$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$.

Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - 1| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then

$$|(2x_n + 1) - 3| = |2(x_n - 1)| = 2|x_n - 1| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $2x_n + 1 \to 3$ as $n \to \infty$. □

Example 2.1.11 *If* $x_n \to -1$ *as* $n \to \infty$*. Prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Proof. Assume that $x_n \to -1$ as $n \to \infty$.

Given $\varepsilon = 1$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n + 1| < 1.$$

Then, $|x_n| - |1| = |x_n| - |-1| \leq |x_n - (-1)| = |x_n + 1| \leq 1$. So, $|x_n| < 2$.

Let $\varepsilon > 0$. By assumption, there is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n + 1| < \frac{\varepsilon}{3}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain

$$\begin{aligned} |(x_n)^2 - 1| &= |(x_n - 1)(x_n + 1)| = |x_n - 1||x_n + 1| \\ &< (|x_n| + 1)\frac{\varepsilon}{3} < (2 + 1)\frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, $(x_n)^2 \to 1$ as $n \to \infty$. □

Example 2.1.12 *Assume that $x_n \to 1$ as $n \to \infty$. Show that*

$$\frac{1}{x_n} \to 1 \text{ as } n \to \infty.$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$.
Given $\varepsilon = \frac{1}{2}$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - 1| < \frac{1}{2}.$$

Then $1 = |1 - x_n + x_n| \leq |1 - x_n| + |x_n| \leq \frac{1}{2} + |x_n|$. So, $\frac{1}{2} \leq |x_n|$. We get $\frac{1}{|x_n|} \leq 2$.
Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - 1| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$. We obtain

$$\left|\frac{1}{x_n} - 1\right| = \left|\frac{1 - x_n}{x_n}\right| \leq \frac{1}{|x_n|} \cdot |x_n - 1| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\frac{1}{x_n} \to 1$ as $n \to \infty$. □

Example 2.1.13 *Assume that $x_n \to 1$ as $n \to \infty$. Show that*

$$\frac{1 + (x_n)^2}{x_n + 1} \to 1 \text{ as } n \to \infty$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$.

Given $\varepsilon = 1$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - 1| < 1.$$

Then $|x_n| - 1 \leq |x_n - 1| \leq 1$. So, $|x_n| \leq 2$. We consider

$$2 = |2 - x_n + x_n| = |1 - x_n + 1 + x_n| \leq |1 - x_n| + |1 + x_n| \leq 1 + |1 + x_n|$$
$$1 \leq |1 + x_n|$$
$$\frac{1}{|1 + x_n|} \leq 1.$$

Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - 1| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$. We obtain

$$\begin{aligned}\left|\frac{1+(x_n)^2}{x_n+1} - 1\right| = \left|\frac{(x_n)^2 - x_n}{x_n+1}\right| = \left|\frac{x_n(x_n-1)}{x_n+1}\right| \\ \leq \frac{|x_n||x_n-1|}{|x_n+1|} = |x_n| \cdot \frac{1}{|x_n+1|} \cdot |x_n - 1| \\ \leq 2 \cdot 1 \cdot \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, Thus, $\dfrac{1+(x_n)^2}{x_n+1} \to 1$ as $n \to \infty$. □

Theorem 2.1.14 *A sequence can have at most one limit.*

Proof. Assume that a sequence $\{x_n\}$ converges to both a and b. We will show that $a = b$ by Corollary 1.1.32. Let $\varepsilon > 0$. By assumption, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \frac{\varepsilon}{2}$$

and

$$n \geq N_2 \quad \text{implies} \quad |x_n - b| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain

$$|a - b| = |(a - x_n) + (x_n - b)| \leq |x_n - a| + |x_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $a - b = 0$ or $a = b$. We conclude that the sequence $\{x_n\}$ can have at most one limit. □

Example 2.1.15 *Show that the limit* $\{(-1)^n\}_{n\in\mathbb{N}}$ *has no limit or does not exist (DNE).*

Proof. Suppose that $(-1)^n \to 1$ as $n \to \infty$. Given $\varepsilon = 1$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |(-1)^n - a| < 1.$$

Since $(-1)^n = \pm 1$, $|1 - a| < 1$ and $|1 + a| = |-1 - a| < 1$. We have

$$2 = |1 + 1| = |(1 - a) + (1 + a)| \leq |1 - a| + |1 + a| < 1 + 1 = 2.$$

It is imposible because $2 < 2$. Thus, $\{(-1)^n\}_{n\in\mathbb{N}}$ has no limit. □

SUBSEQUENCES.

Definition 2.1.16 *By a* ***subsequence*** *of a sequence* $\{x_n\}_{n\in\mathbb{N}}$*, we shall mean a sequence of the form*

$$\{x_{n_k}\}_{k\in\mathbb{N}}, \quad \text{where each } n_k \in \mathbb{N} \text{ and } n_1 < n_2 < n_3 < \ldots$$

Example 2.1.17 *Give examples for two subsequences of the following sequences.*

Sequences	Subsequences
$1, -1, 1, -1, 1, -1, \ldots$	$1, 1, 1, \ldots$
	$-1, -1, -1, \ldots$
$\{n\}_{n\in\mathbb{N}}$	$1, 3, 5, \ldots$
	$2, 4, 6, \ldots$

Consider $\{x_n\}$. We may interest a formula of n_k depending on k. Choose a subsequence $\{x_{n_k}\}$ where $n_k = 2k - 1$ for $k = 1, 2, 3, \ldots$. Then

$$\{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\} = \{x_1, x_3, x_5, \ldots\}.$$

Theorem 2.1.18 *If* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* a *and* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *is any subsequence of* $\{x_n\}_{n\in\mathbb{N}}$*, then*

$$x_{n_k} \text{ converges to } a \text{ as } k \to \infty.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$.
Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \varepsilon.$$

Since $n_k \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \ldots$, it is clear that

$$n_k \geq k \quad \text{for all } k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ such that $k \geq N$. We have $n_k > k \geq N$. So,

$$|x_{n_k} - a| < \varepsilon.$$

Thus, x_{n_k} converges to a as $k \to \infty$. □

Example 2.1.19 *Show that the limit* $\{\cos(n\pi)\}_{n\in\mathbb{N}}$ *has no limit.*

Solution. Choose two subsequences of $\{\cos(n\pi)\}_{n\in\mathbb{N}}$ to be

$$n_k = 2k \quad \text{and} \quad n_k = 2k-1.$$

If $n_k = 2k$, then $\cos(n_k\pi) = \cos(2k\pi) = 1$. So, $\cos(2k\pi) \to 1$ as $k \to \infty$.

If $n_k = 2k-1$, then $\cos(n_k\pi) = \cos(2k-1)\pi = -1$. So, $\cos(2k-1)\pi \to -1$ as $k \to \infty$.

We will see that two subsequences coverges to different limits. Thus, $\{\cos(n\pi)\}_{n\in\mathbb{N}}$ DNE.

BOUNDED SEQUENCES.

Definition 2.1.20 *Let* $\{x_n\}$ *be a sequence of real numbers.*

1. $\{x_n\}$ *is said to be* ***bounded above*** *if and only if*

$$\textit{there is an } M \in \mathbb{R} \textit{ such that } \quad x_n \le M \quad \textit{for all } n \in \mathbb{N}$$

2. $\{x_n\}$ *is said to be* ***bounded below*** *if and only if*

$$\textit{there is an } m \in \mathbb{R} \textit{ such that } \quad m \le x_n \quad \textit{for all } n \in \mathbb{N}$$

3. $\{x_n\}$ *is said to be* ***bounded*** *if and only if it is both above and below or*

$$\textit{there a } K > 0 \textit{ such that } \quad |x_n| \le K \quad \textit{for all } n \in \mathbb{N}$$

Example 2.1.21 *Show that the following sequence is bounded above or bounded below or bounded.*

Sequences	Bounded below	Bounded above	Bounded
$\{n\}_{n\in\mathbb{N}}$	Yes $1 \le n$ for all $n \in \mathbb{N}$	No	No
$\{-n\}_{n\in\mathbb{N}}$	No	Yes $-n \le 1$ for all $n \in \mathbb{N}$	No
$\{(-1)^n\}_{n\in\mathbb{N}}$	Yes $-1 \le (-1)^n$ for all $n \in \mathbb{N}$	Yes $(-1)^n \le 1$ for all $n \in \mathbb{N}$	Yes $\lvert(-1)^n\rvert \le 1$ for all $n \in \mathbb{N}$

Theorem 2.1.22 (**Bounded Convergent Theorem (BCT)**) *Every convergent sequence is bounded.*

Proof. Assume that $x_n \to a$ as $n \to \infty$. Given $\varepsilon = 1$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < 1.$$

Then, $|x_n| - |a| \leq |x_n - a| < 1$. So, $|x_n| \leq 1 + |a|$.
Choose $K = \max\{|x_1|, |x_2|, |x_3|, ..., |x_N|, 1 + |a|\}$. We obtain

$$|x_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Thus, x_{n_k} is bounded. $\square$

Example 2.1.23 *Show that the limit* $\{n\}_{n\in\mathbb{N}}$ *does not exist.*

Solution. Suppose that $\{n\}_{n\in\mathbb{N}}$ converges. By BCT, there is a $K > 0$ such that

$$n = |n| \leq K \quad \text{for all } n \in \mathbb{N} \tag{2.1}$$

Since $K \in \mathbb{R}$, by AP, there is an $N \in \mathbb{N}$ such that $K < N$. By (2.1), $n = N$, we have $N \leq K$. It is imposible because

$$N \leq K < N.$$

Thus, $\{n\}_{n\in\mathbb{N}}$ DNE.

Example 2.1.24 *Assume that* $x_n \to 1$ *as* $n \to \infty$. *Use BCT to prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$. By BCT, there is a $K > 0$ such that

$$|x_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - 1| < \frac{\varepsilon}{K+1}.$$

Let $n \in \mathbb{N}$ such that $n \geq N$, we obtain

$$\begin{aligned} |(x_n)^2 - 1| &= |(x_n - 1)(x_n + 1)| = |x_n - 1||x_n + 1| \\ &< (|x_n| + 1)\frac{\varepsilon}{3} < (K+1)\frac{\varepsilon}{K+1} = \varepsilon. \end{aligned}$$

Thus, $(x_n)^2 \to 1$ as $n \to \infty$. $\square$

Exercises 2.1

1. Prove that the following limit exist.

1.1 $3+\frac{1}{n}$ as $n \to \infty$

1.2 $2\left(1-\frac{1}{n}\right)$ as $n \to \infty$

1.3 $\frac{2n+1}{1-n}$ as $n \to \infty$

1.4 $\frac{n^2-1}{n^2}$ as $n \to \infty$

1.5 $\frac{5+n}{n^2}$ as $n \to \infty$

1.6 $\pi-\frac{3}{\sqrt{n}}$ as $n \to \infty$

1.7 $\frac{n(n+2)}{n^2+1}$ as $n \to \infty$

1.8 $\frac{n}{n^3+1}$ as $n \to \infty$

2. Suppose that x_n is sequence of real numbers that converges to 2 as $n \to \infty$.
   Use Definition 2.1.3, prove that each of the following limit exists.

2.1 $2-x_n \to 0$ as $n \to \infty$

2.2 $3x_n+1 \to 7$ as $n \to \infty$

2.3 $(x_n)^2+1 \to 5$ as $n \to \infty$

2.4 $\frac{1}{x_n-1} \to 1$ as $n \to \infty$

2.5 $\frac{2+x_n^2}{x_n} \to 3$ as $n \to \infty$

3. Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that $\lim\limits_{n\to\infty}(x_n - x_{n+1}) = 0$.

4. If $x_n \to a$ as $n \to \infty$, prove that $x_{n+1} \to a$ as $n \to \infty$.

5. If $x_n \to +\infty$ as $n \to \infty$, prove that $x_{n+1} \to +\infty$ as $n \to \infty$.

6. Prove that $\{(-1)^n\}$ has some subsequences that converge and others that do not converge.

7. Find a convergent subsequence of $n + (-1)^{3n}n$.

8. Suppose that $\{b_n\}$ is a sequence of nonnegative numbers that converges to 0, and $\{x_n\}$ is a real sequence that satisfies $|x_n - a| \le b_n$ for large n. Prove that x_n converges to a.

9. Suppose that $\{x_n\}$ is bounded. Prove that $\frac{x_n}{n^k} \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$.

10. Suppose that $\{x_n\}$ and $\{y_n\}$ converge to same point. Prove that $x_n - y_n \to 0$ as $n \to \infty$

11. Prove that $x_n \to a$ as $n \to \infty$ if and only if $x_n - a \to 0$ as $n \to \infty$.

2.2 Limit theorems

Theorem 2.2.1 (**Squeeze Theorem**) *Suppose that $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are real sequences. If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and there is an $N_0 \in \mathbb{N}$ such that*

$$x_n \leq w_n \leq y_n \quad \textit{for all } n \geq N_0,$$

then $w_n \to a$ as $n \to \infty$.

Proof. Let $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ be real sequences. Assume that $x_n \to a$ and $y_n \to a$ as $n \to \infty$ and there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq w_n \leq y_n \quad \text{for all } n \geq N_0.$$

Let $\varepsilon > 0$. By assumption, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \varepsilon \quad \text{or} \quad a - \varepsilon < x_n < a + \varepsilon$$
$$\text{and}$$
$$n \geq N_2 \quad \text{implies} \quad |y_n - a| < \varepsilon \quad \text{or} \quad a - \varepsilon < y_n < a + \varepsilon.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_0, N_1, N_2\}$. For each $n \geq N$, we obtain

$$a - \varepsilon < x_n \leq w_n \leq y_n < a + \varepsilon.$$

It implies that $|w_n - a| < \varepsilon$. We conclude that $w_n \to a$ as $n \to \infty$. □

Example 2.2.2 *Use the Squeeze Theorem to prove that*

$$\lim_{n\to\infty} \frac{\sin(n^2)}{2^n} = 0.$$

Solution. By the sine fucntion property,

$$-1 \leq \sin(n^2) \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Then, $-\dfrac{1}{2^n} \leq \dfrac{\sin(n^2)}{2^n} \leq \dfrac{1}{2^n}$. From

$$\lim_{n\to\infty} -\frac{1}{2^n} = 0 \text{ and } \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

By the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty} \frac{\sin(n^2)}{2^n} = 0$.

Theorem 2.2.3 *Let $\{x_n\}$, and $\{y_n\}$ be real sequences. If $x_n \to 0$ and $\{y_n\}$ is bounded, then*

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Proof. Let $\{x_n\}$, and $\{y_n\}$ be real sequences. Assume that $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded. Then there is a $K > 0$ such that

$$|y_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n| = |x_n - 0| < \frac{\varepsilon}{K}.$$

Let $n \in \mathbb{N}$. For each $n \geq N$, we obtain

$$|x_n y_n - 0| = |x_n||y_n| < \frac{\varepsilon}{K} \cdot K = \varepsilon.$$

Hence, $x_n y_n \to 0$ as $n \to \infty$. □

Example 2.2.4 *Show that* $\lim_{n\to\infty} \frac{\cos(1+n)}{n^2} = 0$.

Solution. By the cosine fucntion property,

$$|\cos(1+n)| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

So, $\{\cos(1+n)\}$ is bounded. From

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

By Theorem 2.2.3, we conclude that $\lim_{n\to\infty} \frac{\cos(1+n)}{n^2} = 0$.

Theorem 2.2.5 *Let $A \subseteq \mathbb{R}$.*

1. *If A has a finite supremum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \sup A \quad \text{as} \;\; n \to \infty.$$

2. *If A has a finite infimum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \inf A \quad \text{as} \;\; n \to \infty.$$

Proof. Exercise for 1. We will prove 2. Suppose A has a finite infimum. By API, there is $x \in A$ such that

$$\inf A \leq x \leq \inf A + \varepsilon \quad \text{for all } \varepsilon > 0.$$

We construct a sequence $\{x_n\}$ by

$$\begin{aligned}
\varepsilon_1 &= 1, & \exists x_1 \in A \text{ such that} & \quad \inf A \leq x_1 \leq \inf A + 1 \\
\varepsilon_2 &= \frac{1}{2}, & \exists x_2 \in A \text{ such that} & \quad \inf A \leq x_2 \leq \inf A + \frac{1}{2} \\
\varepsilon_3 &= \frac{1}{3}, & \exists x_3 \in A \text{ such that} & \quad \inf A \leq x_3 \leq \inf A + \frac{1}{3} \\
& & \vdots & \\
\varepsilon_n &= \frac{1}{n}, & \exists x_n \in A \text{ such that} & \quad \inf A \leq x_n \leq \inf A + \frac{1}{n}
\end{aligned}$$

Thus, $\{x_n\}$ is a sequence in A and satisfies

$$\inf A \leq x_n < \inf A + \frac{1}{n}$$

By the Squeez Theorem,

$$\begin{gathered}
\lim_{n\to\infty} \inf A \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \left(\inf A + \frac{1}{n} \right) \\
\inf A \leq \lim_{n\to\infty} x_n \leq \inf A
\end{gathered}$$

Therefore, $\lim_{n\to\infty} x_n = \inf A$. $\square$

Theorem 2.2.6 (**Additive Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are real sequences. If* $\{x_n\}$ *and* $\{y_n\}$ *are convergent, then*

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Proof. Assume that $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Let $\varepsilon > 0$. By assumption, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \frac{\varepsilon}{2}$$

and

$$n \geq N_2 \quad \text{implies} \quad |y_n - b| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain

$$|(x_n + y_n) - (a + b)| = |(x_n - a) + (y_n - b)| \leq |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\lim_{n\to\infty}(x_n + y_n) = a + b = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$. □

Theorem 2.2.7 (**Scalar Multiplicative Property**) *Let* $\alpha \in \mathbb{R}$. *If* $\{x_n\}$ *is a convergent sequence, then*

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n.$$

Proof. Assume that $x_n \to a$ as $n \to \infty$.
Let $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. Then $|\alpha| + 1 > |\alpha| \geq 0$. So, $\dfrac{|\alpha|}{|\alpha| + 1} < 1$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{|\alpha| + 1}.$$

Let $n \in \mathbb{N}$. For each $n \geq N$, we obtain

$$|\alpha x_n - \alpha x| = |\alpha||x_n - x| < |\alpha| \cdot \frac{\varepsilon}{|\alpha| + 1} = \frac{|\alpha|}{|\alpha| + 1}\varepsilon < 1 \cdot \varepsilon = \varepsilon.$$

Thus, $\lim_{n\to\infty}(\alpha x_n) = \alpha a = \alpha \lim_{n\to\infty} x_n$.

□

Theorem 2.2.8 (**Multiplicative Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. Then*

$$\lim_{n\to\infty}(x_ny_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

Proof. Assume that $x_n \to a$ and $y_n \to b$ as $n \to \infty$. By BCT, $\{x_n\}$ is bounded, i.e., there is a $K > 0$ such that

$$|x_n| \le K \quad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. By assumption, there are $N_1, N_2 \in \mathbb{N}$ such that

$$n \ge N_1 \quad \text{implies} \quad |x_n - a| < \frac{\varepsilon}{2(|b|+1)}$$
$$\text{and}$$
$$n \ge N_2 \quad \text{implies} \quad |y_n - b| < \frac{\varepsilon}{2K}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \ge N$, we obtain

$$\begin{aligned}|x_ny_n - ab| &= |x_n(y_n - b) + (x_n - a)b| \le |x_n||y_n - b| + |x_n - a||b| \\ &< K \cdot \frac{\varepsilon}{2K} + \frac{\varepsilon}{2(|b|+1)}|b| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot \frac{|b|}{(|b|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot 1 = \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} x_ny_n = ab = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$. □

Theorem 2.2.9 (**Reciprocal Property**) *Suppose that* $\{x_n\}$ *is a convergent sequence.*

$$\lim_{n\to\infty}\frac{1}{x_n} = \frac{1}{\lim_{n\to\infty} x_n}$$

where $\lim_{n\to\infty} x_n \ne 0$ *and* $x_n \ne 0$.

Proof. Assume that $\{x_n\}$ converges to a such that $a \ne 0$.
Given $\varepsilon = \frac{2}{|a|}$. There is an $N_1 \in \mathbb{N}$ such that

$$n \ge N_1 \quad \text{implies} \quad |x_n - a| < \frac{|a|}{2}.$$

Then $|a| = |a - x_n + x_n| \le |x_n - a| + |x_n| \le \frac{|a|}{2} + |x_n|$. So, $\frac{|a|}{2} \le |x_n|$, i.e.,

$$\frac{1}{|x_n|} \le \frac{2}{|a|}.$$

Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that

$$n \ge N_2 \quad \text{implies} \quad |x_n - a| < \frac{|a|^2}{2}\varepsilon.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \ge N$, We obtain

$$\left|\frac{1}{x_n} - \frac{1}{a}\right| = \left|\frac{a - x_n}{ax_n}\right| \le \frac{1}{|x_n|} \cdot \frac{|x_n - a|}{|a|} < \frac{2}{|a|} \cdot \frac{|a|^2}{2|a|}\varepsilon = \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$. □

Theorem 2.2.10 (**Quotient Property**) *Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. Then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

where $\lim\limits_{n\to\infty} y_n \ne 0$ and $y_n \ne 0$.

Proof. The proof of Theorem is result from Multiplicative Property and Reciprocal Property. □

Example 2.2.11 *Find the limit $\lim\limits_{n\to\infty} \frac{n^2 + n - 3}{1 + 3n^2}$.*

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2 + n - 3}{1 + 3n^2} &= \lim_{n\to\infty} \frac{n^2(1 + \frac{1}{n} - \frac{3}{n^2})}{n^2(\frac{1}{n^2} + 3)} \\
&= \lim_{n\to\infty} \frac{1 + \frac{1}{n} - \frac{3}{n^2}}{\frac{1}{n^2} + 3} \\
&= \frac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n} - 3 \lim\limits_{n\to\infty} \frac{1}{n^2}}{\lim\limits_{n\to\infty} \frac{1}{n^2} + \lim\limits_{n\to\infty} 3} \\
&= \frac{1 + 0 - 3(0)}{0 + 3} \\
&= \frac{1}{3}.
\end{aligned}$$

Theorem 2.2.12 (**Comparison Theorem**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. If there is an* $N_0 \in \mathbb{N}$ *such that*

$$x_n \leq y_n \quad \text{for all } n \geq N_0,$$

then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a, b]$ *converges to some point* c*, then* c *must belong to* $[a, b]$.

Proof. Let $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Assume that there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \quad \text{for all } n \geq N_0.$$

Suppose that $\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$, i.e., $a > b$. Then $a - b > 0$. By assumption, there is an $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \frac{a-b}{2}$$
$$\text{and}$$
$$n \geq N_2 \quad \text{implies} \quad |y_n - b| < \frac{a-b}{2}.$$

For each $n \geq \max\{N_0, N_1, N_2\}$, it follows that

$$y_n < b + \frac{a-b}{2} = a - \frac{a-b}{2} < x_n$$

which contradics the assumption. Thus, $a \leq b$.

We conclude by previous proof that if $a \leq x_n \leq b$, $a < c < b$. □

DIVERGENT.

Definition 2.2.13 *Let $\{x_n\}$ be a sequence of real numbers.*

1. *$\{x_n\}$ is said to be **diverge** to $+\infty$, written $x_n \to +\infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n > M.$$

2. *$\{x_n\}$ is said to be **diverge** to $-\infty$, written $x_n \to -\infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = -\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n < M.$$

Example 2.2.14 *Show that* $\lim\limits_{n\to\infty} n = +\infty$

Proof. Let $M \in \mathbb{R}$. By AP, there is an $N \in \mathbb{N}$ such that $M < N$.

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$n \geq N > M.$$

Thus, $\lim\limits_{n\to\infty} n = +\infty$. □

Example 2.2.15 *Prove that* $\lim\limits_{n\to\infty} \dfrac{n^2}{1+n} = +\infty$.

Proof. Let $M \in \mathbb{R}$. By AP, there is an $N \in \mathbb{N}$ such that $M + 1 < N$.

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n - 1 > N - 1$. Since $0 > -1$, $n^2 > n^2 - 1$. We obtain

$$\frac{n^2}{1+n} > \frac{n^2-1}{1+n} = \frac{(n-1)(n+1)}{1+n} = n - 1 > N - 1 > M.$$

Hence, $\lim\limits_{n\to\infty} \dfrac{n^2}{1+n} = +\infty$. □

Example 2.2.16 *Prove that* $\lim\limits_{n\to\infty} \dfrac{4n^2}{1-2n} = -\infty.$

Proof. Let $M \in \mathbb{R}$. By AP, there is an $N \in \mathbb{N}$ such that $-\frac{1}{2}M - \frac{1}{2} < N$. It is equivalent to

$$-1 - 2N < M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. It is clear that $2n - 1 > 0$ and $-2n < -2N$. Since $0 < 1$,

$$-4n^2 < -4n^2 + 1.$$

We obtain

$$\begin{aligned}\frac{4n^2}{1-2n} = \frac{-4n^2}{2n-1} < \frac{-4n^2+1}{2n-1} &= \frac{(1-2n)(1+2n)}{2n-1} \\ &= -1 - 2n < -1 - 2N < M\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{4n^2}{1-2n} = -\infty.$ □

Example 2.2.17 *Suppose that* $\{x_n\}$ *is a real sequence such that* $x_n \to +\infty$ *as* $n \to \infty$*. If* $x_n \neq 0$*, prove that*

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Proof. Assume that $x_n \neq 0$ and $x_n \to +\infty$ as $n \to \infty$.

Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n > \frac{1}{\varepsilon}.$$

From $\dfrac{1}{\varepsilon} > 0$, for all $n \geq N$ it follow that

$$\left|\frac{1}{x_n}\right| = \frac{1}{|x_n|} = \frac{1}{x_n} < \varepsilon.$$

Hence, $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = 0.$ □

Theorem 2.2.18 *Let $\{x_n\}$ and $\{y_n\}$ be a real sequence and $x_n \neq 0$. If $\{y_n\}$ is bounded and $x_n \to +\infty$ or $x_n \to -\infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Proof. Let $\{y_n\}$ be bounded and $x_n \neq 0$. There is a $K > 0$ such that

$$|y_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Case 1. Assume that $x_n \to +\infty$ as $n \to \infty$. Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n > \frac{K}{\varepsilon}.$$

Then $x_n > \frac{K}{\varepsilon} > 0$. It follows that $\frac{1}{|x_n|} = \frac{1}{x_n} < \frac{\varepsilon}{K}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\left|\frac{y_n}{x_n}\right| = |y_n| \cdot \frac{1}{|x_n|} < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Case 2. Assume that $x_n \to -\infty$ as $n \to \infty$. Let $\varepsilon > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < -\frac{K}{\varepsilon}.$$

Since $-\frac{K}{\varepsilon} < 0$, $|x_n| > \frac{K}{\varepsilon} > 0$. It follows that $\frac{1}{|x_n|} < \frac{\varepsilon}{K}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\left|\frac{y_n}{x_n}\right| = |y_n| \cdot \frac{1}{|x_n|} < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

By two cases, we conclude that $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$. □

Example 2.2.19 *Show that $\frac{\sin n}{n} \to 0$ as $n \to \infty$.*

Solution. By property of sine, we have

$$|\sin n| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Since $n \to$ as $n \to \infty$, we obtain by Theorem 2.2.18

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Theorem 2.2.20 *Let $\{x_n\}$ be a real sequence and $\alpha > 0$.*

1. *If $x_n \to +\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = +\infty$.*
2. *If $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = -\infty$.*

Proof. 1. Assume that $x_n \to +\infty$ as $n \to \infty$.

Let $M \in \mathbb{R}$ and $\alpha > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n > \frac{M}{\alpha}.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\alpha x_n > \alpha \cdot \frac{M}{\alpha} = M.$$

Thus, $\lim_{n\to\infty} \alpha x_n = +\infty$.

2. Exercise. □

Theorem 2.2.21 *Let $\{x_n\}$ and $\{y_n\}$ be real sequences. Suppose that $\{y_n\}$ is bounded below and $x_n \to +\infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} (x_n + y_n) = +\infty.$$

Proof. Suppose that $\{y_n\}$ be bounded below and $x_n \to +\infty$ as $n \to \infty$. There is an $m \in \mathbb{R}$ such that

$$m \leq y_n \quad \text{for all } n \in \mathbb{N}.$$

Let $M \in \mathbb{R}$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n > M - m.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$x_n + y_n > (M - m) + m = M.$$

Thus, $\lim_{n\to\infty} (x_n + y_n) = +\infty$. □

Theorem 2.2.22 *Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that*

$$y_n > K \textit{ for some } K > 0 \textit{ and all } n \in \mathbb{N}.$$

It follows that

1. *if $x_n \to +\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = +\infty$*
2. *if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that

$$y_n > K \quad \text{for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

1. Exercise.

2. Assume that $x_n \to -\infty$ as $n \to \infty$. Let $M \in \mathbb{R}$.

Case $M = 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, we obtain

$$x_n \cdot y_n < 0 = M.$$

Case $M > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < -\frac{M}{K} < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, $-y_n < -K < 0$. We obtain

$$x_n \cdot y_n < -\frac{M}{K} \cdot y_n = \frac{M}{K} \cdot (-y_n) < \frac{M}{K} \cdot (-K) = -M < 0 < M.$$

Case $M < 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < \frac{M}{K} < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, $-y_n < -K < 0$. We obtain

$$x_n \cdot y_n < \frac{M}{K} \cdot y_n = \frac{-M}{K} \cdot (-y_n) < \frac{-M}{K} \cdot (-K) = M.$$

Thus, $\lim_{n\to\infty} x_n y_n = -\infty$. ∎

Exercises 2.2

---

1. Prove that each of the following sequences coverges to zero.

   1.1 $x_n = \dfrac{\sin(n^4+n+1)}{n}$

   1.2 $x_n = \dfrac{n}{n^2+1}$

   1.3 $x_n = \dfrac{\sqrt{n}+1}{n+1}$

   1.4 $x_n = \dfrac{n}{2^n}$

   1.5 $x_n = \dfrac{(-1)^n}{n}$

   1.6 $x_n = \dfrac{1+(-1)^n}{2^n}$

2. Find the limit (if it exists) of each of the following sequences.

   2.1 $x_n = \dfrac{2n(n+1)}{n^2+1}$

   2.2 $x_n = \dfrac{1+n-3n^2}{3-2n+n^2}$

   2.3 $x_n = \dfrac{n^3+n+5}{5n^3+n-1}$

   2.4 $x_n = \dfrac{\sqrt{2n^2-1}}{n+1}$

   2.5 $x_n = \sqrt{n+2}-\sqrt{n}$

   2.6 $x_n = \sqrt{n^2+n}-n$

3. Prove that each of the following sequences coverges to $-\infty$ or $+\infty$.

   3.1 $x_n = n^2$

   3.2 $x_n = -n$

   3.3 $x_n = \dfrac{n}{1+\sqrt{n}}$

   3.4 $x_n = \dfrac{n^2+1}{n+1}$

   3.5 $x_n = \dfrac{1-n^2}{n}$

   3.6 $x_n = \dfrac{2^n}{n}$

4. Let $A \subseteq \mathbb{R}$. If A has a finite supremum, then there is a sequence $x_n \in A$ such that

$$x_n \to \sup A \quad \text{as} \quad n \to \infty.$$

5. Prove that given $x \in \mathbb{R}$ there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$.

6. Use the result Excercise 1.2, show that the following

   6.1 Suppose that $0 \le x_1 \le 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n \in \mathbb{N}$.

   If $x_n \to x$ as $n \to \infty$, prove that $x = 0$ or 1.

6.2 Suppose that $x_1 > 0$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$.

If $x_n \to x$ as $n \to \infty$, prove that $x = 2$.

7. Let $\{x_n\}$ be a real sequence and $\alpha > 0$. If $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = -\infty$.

8. Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that $y_n > K$ for some $K > 0$ and all $n \in \mathbb{N}$. Prove that if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$.

9. Let $\{x_n\}$ and $\{y_n\}$ are real sequences. Suuppose that $\{y_n\}$ is bounded above and $x_n \to -\infty$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} (x_n + y_n) = -\infty.$$

10. Interpret a decimal expansion $0.a_1a_2a_3...$ as

$$0.a_1a_2a_3... = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{a_k}{10^k}.$$

Prove that

10.1 $0.5 = 0.4999...$

10.2 $1 = 0.999...$

2.3 Bolzano-Weierstrass Theorem

MONOTONE.

Definition 2.3.1 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.*

1. *$\{x_n\}$ is said to be **increasing** if and only if $x_1 \le x_2 \le x_3 \le \ldots$ or*

$$x_n \le x_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

2. *$\{x_n\}$ is said to be **decreasing** if and only if $x_1 \ge x_2 \ge x_3 \ge \ldots$ or*

$$x_n \ge x_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

3. *$\{x_n\}$ is said to be **monotone** if and only if it is either increasing or decreasing.*

If $\{x_n\}$ is increasing and converges to a, we shall write $x_n \uparrow a$ as $n \to \infty$.
If $\{x_n\}$ is decreasing and converges to a, we shall write $x_n \downarrow a$ as $n \to \infty$.

Example 2.3.2 *Determine whether $\{x_n\}_{n\in\mathbb{N}}$ is increasing or decreasing or NOT both.*

Sequences	Decreasing	Increasing	Monotone
$\{n\}_{n\in\mathbb{N}}$	Yes $1 \le 2 \le 3 \le \ldots$	No	No
$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$	No	Yes $1 \ge \frac{1}{2} \ge \frac{1}{3} \ge \ldots$	No
$\{1\}_{n\in\mathbb{N}}$	Yes $1 \le 1 \le 1 \le \ldots$	Yes $1 \ge 1 \ge 1 \ge \ldots$	Yes
$\{(-1)^n\}_{n\in\mathbb{N}}$	No $-1 \le 1 \ge -1 \le 1 \ge \ldots$	No $-1 \le 1 \ge -1 \le 1 \ge \ldots$	No

---

Theorem 2.3.3 (**Monotone Converegence Theorem (MCT)**) *If $\{x_n\}$ is increasing and bounded above, or if it is decreasing and bounded below, then $\{x_n\}$ has a finite limit.*

Proof. Assume that $\{x_n\}$ is increasing and bounded above. By the Completeness Axiom, the supremum

$$a := \sup\{x_n : n \in \mathbb{N}\} \text{ exists and is finite.}$$

Let $\varepsilon > 0$. By APS, there is an $N \in \mathbb{N}$ such that $a - \varepsilon < x_N \leq a$.

Since $\{x_n\}$ is increasing, $x_N \leq x_n$ for all $n \geq N$. From $x_n \leq a$ for all $n \in \mathbb{N}$. It follows that

$$a - \varepsilon < x_n \leq a \quad \text{for all } n \geq N.$$

So, $-\varepsilon < x_n - a \leq 0 < \varepsilon$. We obtain $|x_n - a| < \varepsilon$. We conclude that $x_n \to a$ as $n \to \infty$.

Exercise for the case that $\{x_n\}$ is decreasing and bounded below. □

Theorem 2.3.4 *If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.*

Proof. Let $|a| < 1$.

Case 1 $a = 0$. Then $a^n = 0$ for all $n \in \mathbb{N}$, and it follows that $a^n \to 0$ as $n \to \infty$.

Case 2 $a \neq 0$. Then $|a| > 0$. We obtain

$$0 < |a|^{n+1} < |a|^n < 1 \quad \text{for all } n \in \mathbb{N}.$$

So, $\{|a|^n\}$ is decreasing and bounded below by 0 . By MCT, $|a|^n \to L$ as $n \to \infty$.

Suppose that $L \neq 0$. Then

$$L = \lim_{n\to\infty} |a|^{n+1} = \lim_{n\to\infty} |a|^n|a| = |a| \lim_{n\to\infty} |a|^n = |a|L.$$

We have $|a| = 1$ which contradics $|a| < 1$. Thus, $L = 0$. □

Example 2.3.5 *Find the limit of* $\left\{\dfrac{3^{n+1}+1}{3^n+2^n}\right\}$.

Solution.

$$\lim_{n\to\infty} \frac{3^{n+1}+1}{3^n+2^n} = \lim_{n\to\infty} \frac{3^n(3+(\frac{1}{3})^n)}{3^n(1+(\frac{2}{3})^n)} = \lim_{n\to\infty} \frac{3+(\frac{1}{3})^2}{1+(\frac{2}{3})^n} = \frac{3+0}{1+0} = 3.$$

Definition 2.3.6 *A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be* ***nested*** *if and only if*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq ... \quad or \quad I_{n+1} \subseteq I_n \text{ for all } n \in \mathbb{N}.$$

Example 2.3.7 *Show that $I_n = [\frac{1}{n}, 1]$ is nested.*

Proof. Let $n \in \mathbb{N}$ and $x \in I_{n+1}$. Then $1 \le x \le \frac{1}{n+1}$. Since $n+1 > n$,

$$1 \le x \le \frac{1}{n+1} < \frac{1}{n}.$$

Then $x \in I_n$. Thus, $I_{n+1} \subseteq I_n$. We conclude that $\{I_n\}_{n\in\mathbb{N}}$ is nested. □

Theorem 2.3.8 (Nested Interval Property) *If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then*

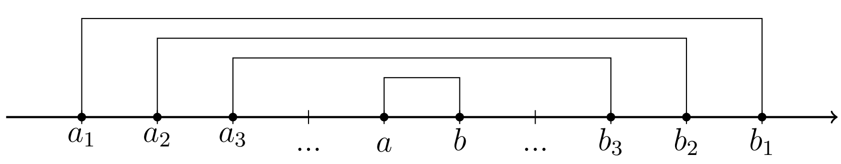
$$E = \bigcap_{n\in\mathbb{N}} I_n := \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

contains at least one number. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E contains exactly one number.

Proof. Let $I_n = [a_n, b_n]$ be nested. Then

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad \text{for all } n \in \mathbb{N}.$$

We obtain $a_1 \le a_2 \le a_3 \le ...$ and $b_1 \ge b_2 \ge b_3 \ge ...$ So, $\{a_n\}$ is increasing and bounded above by a_1 and $\{b_n\}$ is decreasing bounded below by b_1. By MCT, there are a and b such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$.



Since $a_n \le b_n$ for all $n \in \mathbb{N}$, it also follows from the Comparison Theorem that

$$a_n \le a \le b \le b_n.$$

Hence, a number x belongs to I_n for all $n \in \mathbb{N}$ if and only if $a \le x \le b$. We obtain $E = [a, b]$.

Suppose that $|I_n| \to 0$ as $n \to \infty$. Then $b_n - a_n \to 0$ as $n \to \infty$, and we have by Addition Property that $a - b = 0$. In particular, $E = [a, a] = \{a\}$ contain exactly one number. □

Theorem 2.3.9 (**Bolzano-Weierstrass Theorem**) *Every bounded sequence of real numbers has a convergence subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence. Choose $a, b \in \mathbb{R}$ such that

$$x_n \in [a, b] \quad \text{for all } n \in \mathbb{N}.$$

Set $I_0 = [a, b]$. Divide I_0 into two halves, $I_0 = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$. So, at least one of these half intervals contains x_n for infinitely many n. Call it I_1, and choose $n_1 > 1$ such that $x_{n_1} \in I_1$. Notice that

$$|I_1| = \frac{|I_0|}{2} = \frac{b-a}{2}.$$

Suppose that $I_0 \supseteq I_1 \supseteq I_2 \supseteq ... \supseteq I_m$ and natural numbers $n_1 < n_2 < ... < n_m$ have been chosen such that for each $0 \leq k \leq m$,

$$|I_k| = \frac{b-a}{2^k}, \quad x_{n_k} \in I_k, \quad \text{and } x_n \in I_k \quad \text{for infinitely many } n. \tag{2.2}$$

To choose I_{m+1}, divide $I_m = [a_m, b_m]$ into two halves, $I_m = [a_m, \frac{a_m+b_m}{2}] \cup [\frac{a_m+b_m}{2}, b_m]$. So, at least one of these half intervals contains x_n for infinitely many n. Call it I_{m+1}, and choose $n_{m+1} > n_m$ such that $x_{n_{m+1}} \in I_{m+1}$. Since

$$|I_{m+1}| = \frac{|I_m|}{2} = \frac{b_m - a_m}{2^{m+1}},$$

it follows by induction that there is a nested sequence $\{I_k\}_{k \in \mathbb{N}}$ of nonempty closed bounded intervals that satisfy (2.2) for all $k \in \mathbb{N}$. By Nested Interval Property, there is an $x \in \mathbb{R}$ that belongs to I_k for all $k \in \mathbb{N}$. Since $x \in I_k$, we have by (2.2) that

$$0 \leq |x_{n_k} - x| \leq |I_k| \leq \frac{b-a}{2^k} \quad \text{for all } k \in \mathbb{N}.$$

Thus by the Squeeze Theorem, $x_{n_k} \to x$ as $k \to \infty$. □

Exercises 2.3

1. Prove that
$$x_n = \frac{(n^2 + 22n + 65)\sin(n^3)}{n^2 + n + 1}$$
has a convergence sunsequence.

2. If $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ has a finite limit.

3. Suppose that $E \subset \mathbb{R}$ is nonempty bpunded set and $\sup E \notin E$. Prove that there exist a strictly increasing sequence $\{x_n\}$ $(x_1 < x_2 < x_3 < ...)$ that converges to $\sup E$ such that $x_n \in E$ for all $n \in \mathbb{N}$.

4. Suppose that $\{x_n\}$ is a monotone increasing in $\mathbb{R}$ (not necessarily bounded above). Prove that there is extended real number x such that $x_n \to x$ as $n \to \infty$.

5. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Prove that
$$x_n \downarrow 0 \text{ as } n \to \infty \quad \text{and} \quad \frac{x_{n+1}}{x_n} \to \frac{1}{2}, \text{ as } n \to \infty$$

6. If $a > 0$, prove that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. Use the resulte to find the limit of $\{3^{\frac{n+1}{n}}\}$.

7. Let $0 \le x_1 \le 3$ and $x_{n+1} = \sqrt{2x_n + 3}$ for $n \in \mathbb{N}$. Prove that $x_n \uparrow 3$ as $n \to \infty$.

8. Suppose that $x_1 \ge 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $x_n \downarrow 2$ as $n \to \infty$. What happens when $1 \le x_1 < 2$?

9. Prove that
$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

10. Suppose that $x_0 \in \mathbb{R}$ and $x_n = \dfrac{1 + x_{n-1}}{2}$ for $n \in \mathbb{N}$. Prove that $x_n \to 1$ as $n \to \infty$.

11. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove that

    11.1 if $x_n \downarrow 0$, then $x_n > 0$ for all $n \in \mathbb{N}$.

11.2 if $x_n \uparrow 0$, then $x_n < 0$ for all $n \in \mathbb{N}$.

12. Let $0 < y_1 < x_1$ and set

$$x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n}, \quad \text{for } n \in \mathbb{N}$$

12.1 Prove that $0 < y_n < x_n$ for all $n \in \mathbb{N}$.

12.2 Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below.

12.3 Prove that $0 < x_{n+1} - y_{n+1} < \dfrac{x_1 - y_1}{2^n}$ for $n \in \mathbb{N}$

12.4 Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. (the common value is called the arithemetic-geometric mean of x_1 and y_1.)

13. Suppose that $x_0 = 1, y_0 = 0$

$$x_n = x_{n-1} + 2y_{n-1},$$

and

$$y_n = x_{n-1} + y_{n-1}$$

for $n \in \mathbb{N}$. Prove that $x_n^2 - 2y_n^2 = \pm 1$ for $n \in \mathbb{N}$ and

$$\frac{x_n}{y_n} \to \sqrt{2} \quad \text{as} \quad n \to \infty.$$

14. **(Archimedes)** Suppose that $x_0 = 2\sqrt{3}, y_0 = 3,$

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}}, \quad \text{and} \quad y_n = \sqrt{x_n y_{n-1}} \quad \text{for } n \in \mathbb{N}.$$

14.1 Prove that $x_n \downarrow x$ and $y_n \uparrow y$, as $n \to \infty$, for some $x, y \in \mathbb{R}$.

14.2 Prove that $x = y$ and

$$3.14155 < x < 3.14161.$$

(The actual value of x is π.)

2.4 Cauchy sequences

Definition 2.4.1 *A sequence of points $x_n \in \mathbb{R}$ is said to be **Cauchy** if and only if every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n, m \geq N \quad \textit{imply} \quad |x_n - x_m| < \varepsilon.$$

Example 2.4.2 *Show that $\left\{\dfrac{1}{n}\right\}$ is Cauchy.*

Proof. Let $\varepsilon > 0$. By AP, there is an $\mathbb{N} \in \mathbb{N}$ such that $\frac{1}{N} < \dfrac{\varepsilon}{2}$.
Let $m, n \in \mathbb{N}$ such that $n, m \geq N$. Then, $\frac{1}{n} \leq \frac{1}{N}$ and $\frac{1}{m} \leq \frac{1}{N}$. We obtain

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon.$$

Thus, $\left\{\dfrac{1}{n}\right\}$ is Cauchy. □

Theorem 2.4.3 *The sum of two Cauchy sequences is Cauchy.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy. Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that

$$m, n \geq N_1 \quad \text{imply} \quad |x_n - x_m| < \frac{\varepsilon}{2}$$

$$\text{and}$$

$$m, n \geq N_2 \quad \text{imply} \quad |y_n - y_m| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. For $m, n \geq N$, we obtain

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &\leq |x_n - x_m| + |y_n - y_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\{x_n + y_n\}$ is Cauchy. □

Theorem 2.4.4 *If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.*

Proof. Assume that $x_n \to a$ as $n \to \infty$. There are an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \frac{\varepsilon}{2}.$$

Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. We obtain

$$|x_n - x_m| = |(x_n - a) - (x_m - a)| \leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\{x_n\}$ is Cauchy. □

Theorem 2.4.5 (**Cauchy's Theorem**) *Let $\{x_n\}$ be a sequence of real numbers. Then*

$\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges to some point in $\mathbb{R}$.

Proof. Assume that $\{x_n\}$ is Cauchy. Given $\varepsilon = 1$. There is an $N_0 \in \mathbb{N}$ such that

$$|x_m - x_{N_0}| < 1 \quad \text{for all } m \geq N_0.$$

Then, $|x_m| < 1 + |x_{N_0}|$ for $m \geq N_0$. Thus, $\{x_n\}$ is bounded by

$$M = \max\{|x_1|, |x_2|, ..., |x_{N_0-1}|, 1 + |x_{N_0}|\}.$$

By Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ by $x_{n_k} \to a$ as $n \to \infty$. Let $\varepsilon > 0$. There is an $N_1 \in \mathbb{N}$ such that

$$k \geq N_1 \quad \text{implies} \quad |x_{n_k} - a| < \frac{\varepsilon}{2}.$$

Since $\{x_n\}$ is Caucy, thereis an $N_2 \in \mathbb{N}$ such that

$$m, n \geq N_2 \quad \text{implies} \quad |x_m - x_n| < \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_0, N_1, N_2\}$. For each $n \geq N$, we have $n_k \geq N$ since $n_k \geq n$. Then, we obtain

$$|x_n - a| = |(x_n - x_{n_k}) + (x_{n_k} - a)| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{x_n\}$ converges to a.

Coversely, it is clear by Theorem 2.4.4. □

Example 2.4.6 *Prove that any real sequence* $\{x_n\}$ *that satisfies*

$$|x_n - x_{n+1}| \leq \frac{1}{2^n}, \quad n \in \mathbb{N},$$

is convergent.

Proof. Let $\varepsilon > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\dfrac{1}{n} \leq \dfrac{1}{N}$. By the fact that $n < 2^n$ for all $n \in \mathbb{N}$, we get $\dfrac{1}{2^n} < \dfrac{1}{n}$. Suppose that $m > n$. Then $m - n > 0$. So, $1 - \dfrac{1}{2^{m-n}} \leq 1$. We obtain

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} - \cdots + x_{m-1} - x_m| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \\
&= \frac{1}{2^{n-1}} \left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}}\right] \\
&= \frac{1}{2^{n-1}} \sum_{k=1}^{m-n} \frac{1}{2^k} \\
&= \frac{1}{2^n} \left[1 - \frac{1}{2^{m-n}}\right] \\
&\leq \frac{1}{2^n} \\
&< \frac{1}{n} \\
&< \frac{1}{N} < \varepsilon
\end{aligned}$$

Thus, $\{x_n\}$ is Cauchy. Therefore, $\{x_n\}$ is convergent. □

Exercises 2.4

1. Use definition to show that $\{x_n\}$ is Cauchy if

   1.1 $x_n = \dfrac{1}{n^2}$

   1.2 $x_n = \dfrac{n}{n+1}$

2. Prove that the product of two Cauchy sequences is Cauchy.

3. Prove that if $\{x_n\}$ is a sequence that satisfies

$$|x_n| \le \frac{1+n}{1+n+2n^2}$$

   for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.

4. Suppose that $x_n \in \mathbb{N}$ for $n \in \mathbb{N}$. If $\{x_n\}$ is Cauchy prove that there are numbers a and N such that $x_n = a$ for all $n \ge N$.

5. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement:

$$\text{if } n, m \ge N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

   Prove that $\{a_n\}$ converges.

   $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists and is finite.

6. Let $\{x_n\}$ be Cauchy. Prove that $\{x_n\}$ converges if and only if at least one of its subsequence converges.

7. Prove that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k}$ exists and is finite.

8. Let $\{x_n\}$ be a sequence. Suppose that there is an $a > 1$ such that

$$|x_{k+1} - x_k| \le a^{-k}$$

   for all $k \in \mathbb{N}$. Prove that $x_n \to x$ for some $x \in \mathbb{R}$.

9. Show that a sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Chapter 3

Topology on $\mathbb{R}$

3.1 Open sets

Open sets are among the most important subsets of $\mathbb{R}$. A collection of open sets is called a topology, and any property (such as convergence, compactness, or continuity) that can be dened entirely in terms of open sets is called a **topological property**.

Definition 3.1.1 *A set $E \subseteq \mathbb{R}$ is **open** if for every $x \in E$ there exists a $\delta > 0$ such that*

$$(x - \delta, x + \delta) \subseteq E.$$

In other word,

$$\begin{aligned} E \textit{ is open} \quad &\leftrightarrow \quad \forall x \in E\ \exists \delta > 0,\ (x - \delta, x + \delta) \subseteq E \\ &\textit{and} \\ E \textit{ is not open} \quad &\leftrightarrow \quad \exists x \in E\ \forall \delta > 0,\ (x - \delta, x + \delta) \nsubseteq E. \end{aligned}$$

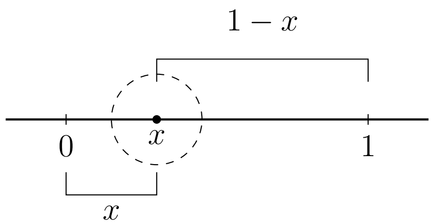
Since the empty set has no element, by definition it imples that $\varnothing$ is open. For $E = \mathbb{R}$, we obatin

$$\forall x \in \mathbb{R}\ \exists \delta > 0,\ (x - \delta, x + \delta) \subseteq \mathbb{R} \text{ is true.}$$

It follows that R is open.

Example 3.1.2 *Show that interval* $(0,1)$ *is open.*

Proof. Let $x \in (0,1)$. Choose $\delta = \min\left\{\frac{x}{2}, \frac{1-x}{2}\right\}$.



We obtain $(x-\delta, x+\delta) \subseteq (0,1)$. Hence, $(0,1)$ is open. □

Theorem 3.1.3 *Intervals* (a,b), (a,∞) *and* $(-\infty,b)$ *are open.*

Proof.

1. Let $x \in (a,b)$. Choose $\delta = \min\left\{\frac{x-a}{2}, \frac{b-x}{2}\right\}$. We obtain $(x-\delta, x+\delta) \subseteq (a,b)$. Hence, (a,b) is open.
2. Let $x \in (a,\infty)$. Choose $\delta = \frac{x-a}{2}$. We obtain $(x-\delta, x+\delta) \subseteq (a,\infty)$. Hence, (a,∞) is open.
3. Let $x \in (-\infty,b)$. Choose $\delta = \frac{b-x}{2}$. We obtain $(x-\delta, x+\delta) \subseteq (-\infty,b)$. Hence, $(-\infty,b)$ is open.

□

Example 3.1.4 *Show that* $[0,1)$ *is not open.*

Proof. Suppose that $[0,1)$ is open. Given $x=0$, there is a $\delta > 0$ such that

$$(-\delta, \delta) \subseteq [0,1).$$

Since $-\delta < -\frac{\delta}{2} < 0$, $-\frac{\delta}{2} \in (-\delta,\delta)$. It implies that $-\frac{\delta}{2} \in [0,1)$ which is imposible. □

Theorem 3.1.5 *Let A and B be open. Prove that $A \cup B$ and $A \cap B$ are open.*

Proof. Let A and B be open.

1. Let $x \in A \cup B$. Then $x \in A$. There is a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq A$. Since $A \subseteq A \cup B$, $(x - \delta, x + \delta) \subseteq A \cup B$. Thus, $A \cup B$ is open.

2. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. There are $\delta_1, \delta_2 > 0$ such that

$$(x - \delta_1, x + \delta_1) \subseteq A \text{ and } (x - \delta_2, x + \delta_2) \subseteq B.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. We obtain $(x - \delta, x + \delta) \subseteq A \cap B$. Thus, $A \cap B$ is open.

□

Theorem 3.1.6 *Let $A_1, A_2, .., A_n$ be open sets. Then*

1. $\displaystyle\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is open.*
2. $\displaystyle\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is open.*

Proof. Excercise □

NEIGHBORHOOD.

Next, we introduce the notion of the neighborhood of a point, which often gives clearer, but equivalent, descriptions of topological concepts than ones that use open intervals.

Definition 3.1.7 *A set $U \subseteq \mathbb{R}$ is a* ***neighborhood*** *of a point $x \in \mathbb{R}$ if*

$$(x-\delta, x+\delta) \subseteq U \quad \text{for some } \delta > 0.$$

For example $x = 1$, we have $(0, 2)$, $[0, 2]$ and $[0, 2)$ to be neighborhoods of 1.

Theorem 3.1.8 *A set $E \subseteq \mathbb{R}$ is open if every $x \in E$ has a neighborhood U such that $U \subseteq E$.*

Proof. If every $x \in E$ has a neighborhood U such that $U \subseteq E$, then there is a $\delta > 0$ such that

$$(x-\delta, x+\delta) \subseteq U \subseteq E.$$

Hence, $E \subseteq \mathbb{R}$ is open . □

Theorem 3.1.9 *A sequence $\{x_n\}$ of real numbers converges to a limit $x \in \mathbb{R}$ if and only if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.*

Proof. Assume that $x_n \to x$ as $n \to \infty$. Let U be a neighborhood of x. There is a $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \subseteq U.$$

By assumption, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \varepsilon$.
It follows that $x - \varepsilon < x_n < x + \varepsilon$. Thus, $x_n \in (x-\varepsilon, x+\varepsilon) \subseteq U$ for all $n \geq N$.
Conversely, assume that for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. Let $\varepsilon > 0$. Fixed x. Then $(x-\varepsilon, x+\varepsilon)$ is a neighborhood of x.
By assumption, there exists $N \in \mathbb{N}$ such that $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n > N$. We have

$$|x_n - a| < \varepsilon \quad \text{for all } n \geq N.$$

Therefore, $x_n \to x$ as $n \to \infty$. □

Exercises 3.1

1. Show that interval $[a,b]$, $[a,b)$ and $(a,b]$, are not open.

2. Show that interval $[a,\infty)$ and $(-\infty,b]$ are not open.

3. Give two neighborhoods of $x=2$.

4. Let A and B be subsets of $\mathbb{R}$. Suppose that A and B are open.

   Determine whether $A\backslash B$ is open.

5. Let $U \subseteq \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

6. Let $A_1, A_2, .., A_n$ be open sets. Prove that

   6.1 $\bigcup\limits_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ is open.

   6.2 $\bigcap\limits_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ is open.

7. Find a sequence I_n of bounded, and open interval that

$$I_{n+1} \subset I_n \text{ for each } n \in \mathbb{N} \text{ and } \bigcap_{n=1}^{\infty} I_n = \varnothing.$$

3.2 Closed sets

Definition 3.2.1 *A set $F \subseteq \mathbb{R}$ is* ***closed*** *if*

$$F^c = \mathbb{R} \backslash F = \{x \in \mathbb{R} : x \notin F\} \text{ is open.}$$

Since $\varnothing^c = \mathbb{R}$ and $\mathbb{R}^c = \varnothing$ ($\varnothing$ and $\mathbb{R}$ are open), $\varnothing$ and $\mathbb{R}$ are closed sets.

Example 3.2.2 *Show that interval $[0, 1]$ is closed.*

Solution. Consider $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$. By Theorem 3.1.3 and 3.1.5, we obtain

$$(-\infty, 0) \cup (1, \infty) \text{ is open.}$$

We conclude that $[0, 1]$ is closed.

Example 3.2.3 *Show that $[0, 1)$ is neither open nor closed.*

Solution. Consider $[0, 1)^c = (-\infty, 0) \cup [1, \infty)$. Choose $x = 1$. Then

$$(1 - \delta, 1 + \delta) \nsubseteq (-\infty, 0) \cup [1, \infty) \text{ for all } \delta > 0.$$

So, $(-\infty, 0) \cup [1, \infty)$ is not open. We conclude that $[0, 1)$ is neither open nor closed.

Theorem 3.2.4 *Let A and B be closed. Prove that $A \cup B$ and $A \cap B$ are closed.*

Proof. Let A and B be closed. Then A^c and B^c are open. By Theorem 3.1.5, it implies that

$$A^c \cap B^c \text{ and } A^c \cup B^c \text{ are open.}$$

Since $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$,

$$(A \cup B)^c \text{ and } (A \cap B)^c \text{ are open.}$$

We conclude that $A \cup B$ and $A \cap B$ are closed. □

Theorem 3.2.5 *Let $A_1, A_2, .., A_n$ be closed sets. Then*

1. $\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is closed.*

2. $\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is closed.*

Proof. Let $A_1, A_2, .., A_n$ be closed sets. Then $A_1^c, A_2^c, .., A_n^c$ are open. We consider

$$\left(\bigcup_{k=1}^{n} A_k\right)^c = (A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$$
$$\left(\bigcap_{k=1}^{n} A_k\right)^c = (A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup A_n^c$$

By theorem 3.1.6, it follows that

$$\left(\bigcup_{k=1}^{n} A_k\right)^c \text{ and } \left(\bigcap_{k=1}^{n} A_k\right)^c \text{ are open.}$$

The proof of Theorem is complete. □

Exercises 3.2

1. Show that interval $[a, b]$, $[a, \infty)$ and $(-\infty, b]$ are closed.

2. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

3. Show that every closed interval I is a closed set.

4. Is $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{n+1}{n}\right)$ open or closed ?

5. Is $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \frac{n-1}{n}\right]$ open or closed ?

6. Suppose, for $n \in \mathbb{N}$, the intervals $I_n = [a_n, b_n]$ are such that $I_{n+1} \subset I_n$. If

$$a = \sup\{a_n : n \in \mathbb{N}\} \quad \text{and} \quad b = \inf\{b_n : n \in \mathbb{N}\},$$

   show that $\displaystyle\bigcap_{n=1}^{\infty} I_n = [a, b]$.

7. Find a sequence I_n of closed interval that $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$ and $\displaystyle\bigcap_{n=1}^{\infty} I_n = \varnothing$.

8. Suppose that $U \subseteq \mathbb{R}$ is a nonempty open set. For each $x \in U$, let

$$J_x = \big(x - \varepsilon, x + \delta\big),$$

   where the union is taken over all $\varepsilon > 0$ and $\delta > 0$ such that $(x - \varepsilon, x + \delta) \subset U$.

   8.1 Show that for every $x, y \in U$, either $J_x \cap J_y = \varnothing$, or $J_x = J_y$.

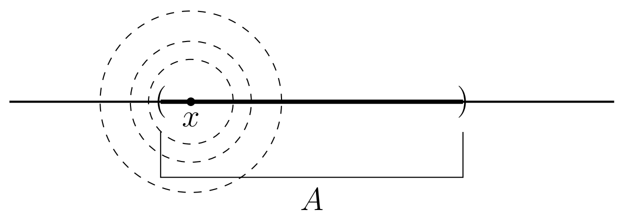
   8.2 Show that $U = \displaystyle\bigcup_{x \in B} J_x$, where $B \subseteq U$ is either finite or countable.

3.3 Limit points

Definition 3.3.1 *A point $x \in \mathbb{R}$ is called a* ***limit point*** *of a set $A \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there exists $a \in A$, $a \neq x$, such that $a \in (x-\varepsilon, x+\varepsilon)$ or*

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A \neq \varnothing.$$

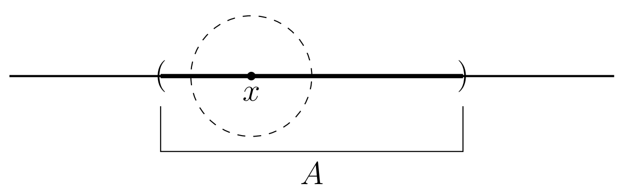
We denote the set of all limit points of a set A by A'.



Definition 3.3.2 *Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is an* ***interior point*** *of A if there exists an $\delta > 0$ such that*

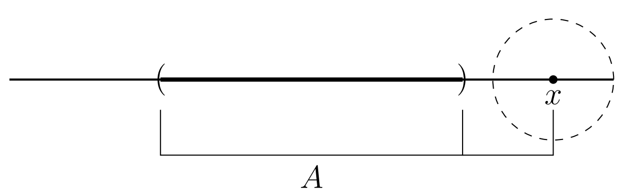
$$(x-\delta, x+\delta) \subseteq A.$$

The set of all interior points of A is called the interior of A, denoted A°.



Definition 3.3.3 *Suppose $A \subseteq \mathbb{R}$. A point $x \in A$ is called an* ***isolated point*** *of A if there exists an $\delta > 0$ such that*

$$A \cap (x-\delta, x+\delta) = \{x\}.$$



Example 3.3.4 *Fill the blanks of the following table.*

Set	Set of limit points	Set of interior points	Set of isolated points
$[0,1]$	$[0,1]$	$(0,1)$	$\varnothing$
$(0,1)$	$[0,1]$	$(0,1)$	$\varnothing$
$[0,1)$	$[0,1]$	$(0,1)$	$\varnothing$
$(0,1]\cup\{3\}$	$[0,1]$	$(0,1)$	$\{3\}$
$\{1\}$	$\varnothing$	$\varnothing$	$\{1\}$
$\mathbb{N}$	$\varnothing$	$\varnothing$	$\mathbb{N}$
$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$

Example 3.3.5 *Show that* 0 *is a limit point of* $(0,1)$.

Proof. Let $\varepsilon > 0$. By AP, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$.
Choose $a = \dfrac{1}{N+1}$. We have,

$$\frac{1}{N+1} < \frac{1}{N} < \varepsilon.$$

It implies that $\dfrac{1}{N+1} \in (-\varepsilon, \varepsilon)$. Since $N+1 > 1$, $0 < \dfrac{1}{N+1} < 1$. We obatin

$$\frac{1}{N+1} \in (0,1).$$

We obtain

$$[(-\varepsilon, 0) \cup (0, \varepsilon)] \cap (0,1) \neq \varnothing.$$

Thus, 0 is a limit point of $(0,1)$. ∎

Theorem 3.3.6 *Let A and B be sets. If $A \subseteq B$, then $A' \subseteq B'$.*

Proof. Let A and B be sets such that $A \subseteq B$. Let $x \in A'$. Then, for all $\varepsilon > 0$, we obtain

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A \neq \varnothing.$$

Since $A \subseteq B$,

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap B \neq \varnothing.$$

So, $x \in B'$. We conclude that $A' \subseteq B'$. □

Theorem 3.3.7 *Let A be a closed subset of $\mathbb{R}$. Then $A' \subseteq A$.*

Proof. Assume that A is closed. Then A^c is open.

Let $x \in A'$ or x be a limit point of A.

Suppose that $x \notin A$. Then $x \in A^c$. There is an $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \subseteq A^c.$$

It follows that $(x-\varepsilon, x+\varepsilon) \cap A = \varnothing$. Since $x \notin A$,

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A = \varnothing.$$

So, x is not a limit point of A which is imposible. Thus, $x \in A$. □

CLOSURE.

Definition 3.3.8 *Given a set $A \subseteq R$, the set $\bar{A} = A \cup A'$ is called the* ***closure*** *of A.*

Example 3.3.9 *Fill the blanks of the following table.*

Set	Set of limit points	Closure
$[0,1]$	$[0,1]$	$[0,1]$
$(0,1)$	$[0,1]$	$[0,1]$
$[0,1)$	$[0,1]$	$[0,1]$
$(0,1] \cup \{3\}$	$[0,1]$	$[0,1] \cup \{3\}$
$\{1\}$	$\varnothing$	$\{1\}$
$\mathbb{N}$	$\varnothing$	$\mathbb{N}$
$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{R}$

Theorem 3.3.10 *Let A and B be subsets of $\mathbb{R}$. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Proof. Let A and B be sets such that $A \subseteq B$. By Theorem 3.3.6, it implies that $A' \subseteq B'$. We conclude that $\bar{A} = A \cup A' \subseteq B \cup B' = \bar{B}$. □

Theorem 3.3.11 *Let $A \subseteq \mathbb{R}$. Then $\bar{A}$ is closed.*

Proof. Let $x \in (\bar{A})^c = (A \cup A')^c$. Then $x \notin A$ and $x \notin A'$. There is an $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap A = [(x - \varepsilon, x) \cup (x, x + \varepsilon)] \cap A = \varnothing.$$

Since $x \notin A$, $(x - \varepsilon, x + \varepsilon) \cap A = [(x - \varepsilon, x) \cup (x, x + \varepsilon)] \cap A$. Use the fact that $A \subseteq \bar{A}$, we obtain

$$(x - \varepsilon, x + \varepsilon) \cap \bar{A} = \varnothing.$$

So, $(x - \varepsilon, x + \varepsilon) \subseteq (\bar{A})^c$. Thus, $(\bar{A})^c$ is open. We conclude that $\bar{A}$ is closed. □

Theorem 3.3.12 *Let $A \subseteq \mathbb{R}$. Then A is closed if and only if $A = \bar{A}$.*

Proof. Assume that A is closed. By Theorem 3.3.7, $A' \subseteq A$. It follows that

$$\bar{A} = A \cup A' \subseteq A.$$

From definition of closer, $A \subseteq A \cup A' = \bar{A}$. Thus, $A = \bar{A}$.

Coversely, assume that $A = \bar{A}$. By Theorem 3.3.11, $\bar{A}$ is closed. Hence, A is also closed. □

Theorem 3.3.13 *A set $F \subseteq \mathbb{R}$ is closed if and only if*

the limit of every convergent sequence in F belongs to F.

Proof. Let F be a closed set. Assume that $\{x_n\}$ is a sequence in F. We will prove by contradiction.

Assume that $x_n \to a$ as $n \to \infty$ and $a \notin F$. Then $a \in F^c$.

Since F^c is open, there $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq F^c$. So,

$$(a - \delta, a + \delta) \cap F = \varnothing \tag{3.1}$$

From $x_n \to a$ as $n \to \infty$, $(\varepsilon = \delta)$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \delta.$$

Then $x_n \in (a - \delta, a + \delta)$. But $x_n \in F$, this is contradiction to (3.1). Thus, $a \in F$.

Coversely, we will prove in Excercise. □

Exercises 3.3

1. Identify the limit points, interior point and isolated points of the following sets:

   1.1 $A = (0, 1) \cup \{3\}$

   1.2 $A = [0, 1]^c$

   1.3 $A = [1, \infty)$

   1.4 $A = (0, 1) \cup [3, 4]$

   1.5 $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

   1.6 $A = [0, 1] \cap \mathbb{Q}$

2. Find A', A° and $\bar{A}$ where

   2.1 $A = (0, 1)$

   2.2 $A = [0, 1]$

   2.3 $A = [0, \infty)$

   2.4 $A = (0, 1) \cup \{2, 3\}$

   2.5 $A = \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\}$

   2.6 $A = \mathbb{Q}$

3. Let A and B be two subset of $\mathbb{R}$. Show that $(A \cup B)' = A' \cup B'$.

4. Let A and B be two subset of $\mathbb{R}$. Determine whether

   4.1 $(A \cap B)' = A' \cap B'$

   4.2 $\overline{A \cup B} = \bar{A} \cup \bar{B}$

   4.3 $\overline{A \cap B} = \bar{A} \cap \bar{B}$

   4.4 $(A \cup B)^\circ = A^\circ \cup B^\circ$

   4.5 $(A \cap B)^\circ = A^\circ \cap B^\circ$

   4.6 if $\bar{A} \subseteq \bar{B}$, then $A \subseteq B$.

5. Prove that A° is open.

6. Prove that A is open if and only if $A = A^\circ$.

7. Suppose x is a limit point of the set A. Show that for every $\varepsilon > 0$, the set
$$(x - \varepsilon, x + \varepsilon) \cap A \text{ is infinite.}$$

8. Suppose that $A_k \subseteq \mathbb{R}$ for each $k \in \mathbb{N}$, and let $B = \bigcup_{k=1}^{\infty} A_k$. Show that $\bar{B} = \bigcup_{k=1}^{\infty} \bar{A}_k$.

9. If the limit of every convergent sequence in F belongs to $F \subseteq \mathbb{R}$, prove that F is closed.

Chapter 4

Limit of Functions

4.1 Limit of Functions

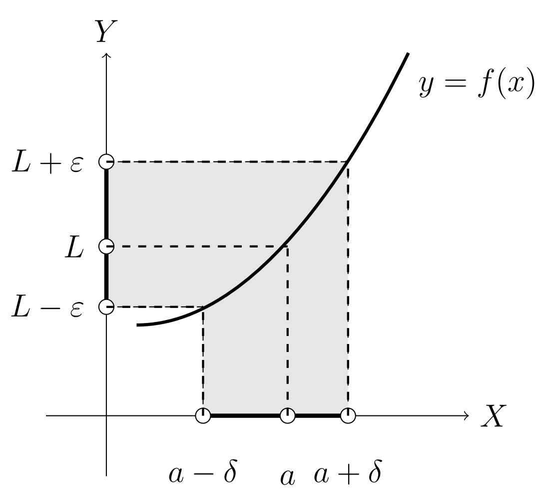
Definition 4.1.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a limit point of E. Then $f(x)$ is said to* ***converge*** *to L, as x* ***approaches*** *a, if and only if*
for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in E$,

$$0 < |x - a| < \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we write

$$\lim_{x \to a} f(x) = L \quad \textit{or} \quad f(x) \to L \textit{ as } x \to a.$$

and call L the ***limit*** *of $f(x)$ as x approaches a.*



Example 4.1.2 *Suppose that* $f(x) = 2x + 1$. *Prove that*

$$\lim_{x \to 1} f(x) = 3.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2} > 0$. Let $x \in \mathbb{R}$ such that $0 < |x - 1| < \delta$. We obtain

$$|f(x) - 3| = |(2x + 1) - 3| = |2(x - 1)| = 2|x - 1| < 2\delta = \varepsilon.$$

Thus, $f(x) \to 3$ as $x \to 1$. □

Example 4.1.3 *Let* $f(x) = \sqrt{x^2}$ *where* $x \in \mathbb{R}$. *Prove that* $f(x) \to 0$ *as* $x \to 0$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon > 0$. Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$. We obtain

$$|f(x) - 0| = |\sqrt{x^2} - 0| = |x| < \varepsilon.$$

Thus, $\sqrt{x^2} \to 0$ as $x \to 0$. □

Example 4.1.4 *Prove that*

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon > 0$. Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$.
Use the property of cosine that

$$\left|\cos\left(\frac{1}{x}\right)\right| \leq 1 \text{ for all } x \neq 0.$$

We obtain

$$\left|x \cos\left(\frac{1}{x}\right) - 0\right| = \left|x \cos\left(\frac{1}{x}\right)\right| = |x| \left|\cos\left(\frac{1}{x}\right)\right| \leq |x| \cdot 1 = |x| < \delta = \varepsilon.$$

Thus, $x \cos\left(\dfrac{1}{x}\right) \to 0$ as $x \to 0$. □

Example 4.1.5 *Prove that*

$$\lim_{x\to 3} x^2 = 9.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$. Let $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$.
Then $0 < |x-3| < 1$. By Triangle inequality, $|x| - 3 < |x-3| < 1$. So, $|x| < 4$. We obtain

$$|x^2 - 9| = |(x+3)(x-3)| = |x+3||x-3| \le (|x| + |3|)\delta < (4+3)\frac{\varepsilon}{7} = \varepsilon.$$

Thus, $\sqrt{x} \to 0$ as $x \to 0$. □

Example 4.1.6 *Prove that* $f(x) = \frac{1}{x} \to 1$ *as* $x \to 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$. Let $x \in \mathbb{R}\backslash\{0\}$ such that $0 < |x-1| < \delta$.
Then $0 < |x-1| < \frac{1}{2}$. By Triangle inequality,

$$1 = |1 - x + x| \le |1-x| + |x| < \frac{1}{2} + |x|.$$

So, $|x| > \frac{1}{2}$. It follows that $\frac{1}{|x|} < 2$. We obtain

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| = \frac{1}{|x|} \cdot |x-1| < 2\delta < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $f(x) \to \frac{1}{x}$ as $x \to 1$. □

Theorem 4.1.7 (Limit of Constant function) *The limit of a constant function is equal to the constant.*

Proof. Let K be a constant. Define $f(x) = K$ for all $x \in \mathbb{R}$.
Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Whatever a positive δ, we obtain for all $x \in \mathbb{R}$,

$$0 < |x-a| < \delta \quad \text{implies} \quad |K - K| = 0 < \varepsilon.$$

We conclude that $\lim_{x\to a} K = K$. □

Theorem 4.1.8 (**Limit of Linear function**) *Let m and c be constant such that $f(x) = mx + c$ for all $x \in \mathbb{R}$. Then*

$$\lim_{x \to a}(mx + c) = ma + c.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{|m| + 1} > 0$. Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
We obtain by $\dfrac{|m|}{|m| + 1} < 1$ that

$$\begin{aligned}|f(x) - (ma + c)| &= |(mx + c) - (ma + c)| = |m(x - a)| \\ &= |m||x - a| < |m|\delta = |m| \cdot \frac{\varepsilon}{|m| + 1} < 1 \cdot \varepsilon = \varepsilon.\end{aligned}$$

Thus, $f(x) \to (ma + c)$ as $x \to a$. □

Theorem 4.1.9 *Let $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ be a limit point of E. If*

$$f(x) = g(x) \text{ for all } x \in E \setminus \{a\} \quad \text{and} \quad f(x) \to L \text{ as } x \to a,$$

then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Proof. Assume that $f(x) = g(x)$ for all $x \in E \setminus \{a\}$ and $f(x) \to L$ as $x \to a$.
Let $\varepsilon > 0$. There is a $\delta > 0$.

$$\forall x \in E,\ 0 < |x - a| < \delta \ \to \ |f(x) - L| < \varepsilon.$$

From $0 < |x - a| < \delta$, it implies that $x \neq a$. So, $f(x) = g(x)$ on the condition. We obtain

$$\forall x \in E,\ 0 < |x - a| < \delta \ \to \ |g(x) - L| < \varepsilon.$$

Thus, $g(x) \to L$ as $x \to a$. □

Example 4.1.10 *Prove that $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a limit as $x \to 1$.*

Solution. We see that $g(x) = x + 1$. We have

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 = g(x) \quad \text{for all } x \neq 1$$

By Theorem 4.1.9, it follows that

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1}(x + 1) = 2.$$

Theorem 4.1.11 (**Sequential Characterization of Limit (SCL)**) *Let* $E \subseteq \mathbb{R}$ *and* $f : E \to \mathbb{R}$ *be a function and let* $a \in \mathbb{R}$ *be a limit point of* E*. Then*

$$\lim_{x \to a} f(x) = L \quad \textit{exists}$$

if and only if $f(x_n) \to L$ *as* $n \to \infty$ *for every sequence* $x_n \in E \backslash \{a\}$ *that converges to* a *as* $n \to \infty$*.*

Proof. Assume that the limit of $f(x)$ exists and equals to L and assume that a sequence $x_n \in E \backslash \{a\}$ that converges to a as $n \to \infty$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that for all $x \in E$,

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon. \tag{4.1}$$

There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \delta.$$

Since $x_n \neq \{a\}$ and $|x_n - a| < \delta$ for all $n \geq N$, we obtain by (4.1)

$$|f(x_n) - L| < \varepsilon \quad \text{for all } n \geq N.$$

Coversely, assume that $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in E \backslash \{a\}$ that converges to a as $n \to \infty$. Suppose that $f(x)$ does not converge to L as x approaches to a.
There is an $\varepsilon_0 > 0$ such that

$$\forall \delta > 0,\ 0 < |x - a| < \delta \ \text{ and } \ |f(x) - L| \geq \varepsilon_0. \tag{4.2}$$

Choose $\delta = \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Then $0 < |x - a| < \dfrac{1}{n}$. By Squeeze Theorem, $x_n \to a$ as $n \to \infty$.
By assumption, $f(x_n) \to L$ as $n \to \infty$, i.e., there $N \in \mathbb{N}$

$$n \geq N \quad \text{implies} \quad |f(x) - L| < \varepsilon_0$$

which contradics (4.2). Therefore, $f(x)$ converges to L as x approaches to a.

□

Example 4.1.12 *Prove that*

$$f(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has no limit as $x \to 0$.

Solution. Choose two sequence as follow

$$\begin{aligned} x_n &= \frac{1}{2n\pi} \to 0 \quad \text{and} \quad f(x_n) = \cos(2n\pi) \to 1, \\ y_n &= \frac{1}{(2n-1)\pi} \to 0 \quad \text{and} \quad f(y_n) = \cos(2n-1)\pi \to -1. \end{aligned}$$

Then $f(x_n)$ and $f(y_n)$ converge to distinct limits. By SCL, we conclude that f has no limit as $x \to 0$.

Next, we will use the SCL together Theorems of limit for addition, mutiplication, scalar multiplication and quotient in order to proof Theorem 4.1.13.

Theorem 4.1.13 *Let* $\alpha \in \mathbb{R}$, $E \subseteq \mathbb{R}$ *and* $f, g : E \to \mathbb{R}$ *be functions and let* $a \in \mathbb{R}$ *be a limit point of* E. *If* $f(x)$ *and* $g(x)$ *converge as* x *approaches* a, *then so do*

$$(f+g)(x),\ (\alpha f)(x),\ (fg)(x) \text{ and } (\tfrac{f}{g})(x).$$

In fact,

1. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x)$
3. $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *when the limit of* $g(x)$ *is nonzero.*

Example 4.1.14 *Show that* $\lim_{x\to a} x^2 = a^2$ *fo all* $a \in \mathbb{R}$.

Solution. Use Theorem 4.1.13 to give

$$\lim_{x\to a} x^2 = \lim_{x\to a} x \cdot x = \lim_{x\to a} x \cdot \lim_{x\to a} x = a \cdot a = a^2.$$

Theorem 4.1.15 *Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is a function. Let $a \in \mathbb{R}$ be a limit point of E. Then,*

$$\lim_{x\to a} |f(x)| = 0 \quad \textit{if and only if} \quad \lim_{x\to a} f(x) = 0.$$

Proof. Exercise. □

Theorem 4.1.16 (**Squeeze Theorem for Functions**) *Suppose that $E \subseteq \mathbb{R}$ and $f, g, h : E \to \mathbb{R}$ are functions. Let $a \in \mathbb{R}$ be a limit point of E. If*

$$g(x) \le f(x) \le h(x) \quad \textit{for all } x \in E\backslash\{a\},$$

and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then the limit of $f(x)$ exists, as $x \to a$ and

$$\lim_{x\to a} f(x) = L.$$

Proof. Use SCL and the Squeeze Thorem (Theorem 2.2.1). □

Corollary 4.1.17 *Suppose that $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ are functions. Let $a \in \mathbb{R}$ be a limit point of E and $M > 0$. If*

$$|g(x)| \le M \quad \textit{for all } x \in E\backslash\{a\} \quad \textit{and} \quad \lim_{x\to a} f(x) = 0,$$

then

$$\lim_{x\to a} f(x)g(x) = 0.$$

Proof. Assume that $|g(x)| \le M$ for all $x \in E\backslash\{a\}$ and $\lim_{x\to a} f(x) = 0$.
Case $f(x) = 0$. Then $f(x)g(x) = 0$. It follows that $\lim_{x\to a} f(x)g(x) = 0$.
Case $f(x) \ne 0$. Then $|f(x)| > 0$. So, $\lim_{x\to a} M|f(x)| = 0$. We obtain

$$0 \le |g(x)f(x)| = |g(x)||f(x)| \le M|f(x)|.$$

By the Squeeze Theorem for Functions, it imlies that $\lim_{x\to a} |g(x)f(x)| = 0$.
From Theorem 4.1.15, we conclude that $\lim_{x\to a} f(x)g(x) = 0$. □

Example 4.1.18 *Show that* $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

Solution. By property of sine,

$$\left|\cos\left(\frac{1}{x}\right)\right| \leq 1 \text{ for all } x \neq 0.$$

We have $\lim_{x\to 0} x = 0$. By Corollary 4.1.17, $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$.

Theorem 4.1.19 (**Comparision Theorem for Functions**) *Suppose that* $E \subseteq \mathbb{R}$ *and* $f, g : E \to \mathbb{R}$ *are functions. Let* $a \in \mathbb{R}$ *be a limit point of* E*. If* f *and* g *have a limit as* x *approaches* a *and*

$$f(x) \leq g(x), \quad x \in E\backslash\{a\},$$

then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Proof. Use SCL together the Comparison Theorem (Theorem 2.2.12), we will this theorem. □

Exercises 4.1

1. Use Definition 4.1.1, prove that each of the following limit exists.

   1.1 $\lim_{x\to 1} x^2 = 1$

   1.2 $\lim_{x\to 2} x^2 - x + 1 = 3$

   1.3 $\lim_{x\to -1} x^3 + 1 = 0.$

   1.4 $\lim_{x\to 0} \dfrac{x-1}{x+1} = -1$

2. Decide which of the following limit exist and which do not.

   2.1 $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$

   2.2 $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

   2.3 $\lim_{x\to 0} \tan\left(\dfrac{1}{x}\right)$

3. Evaluate the following limit using result from this section.

   3.1 $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^3 - x}$

   3.2 $\lim_{x\to \sqrt{\pi}} \dfrac{\sqrt[3]{\pi - x^2}}{x + \pi}$

   3.3 $\lim_{x\to 0} x\sin\left(\dfrac{1}{x^2}\right)$

   3.4 $\lim_{x\to 0} x^2\cos\left(\dfrac{1}{x}\right)$

4. Prove that $\lim_{x\to 0} x^n \sin\left(\dfrac{1}{x}\right)$ exists for all $n \in \mathbb{N}$.

5. Show that $\lim_{x\to a} x^n = a^n$ fo all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

6. Prove that $\lim_{x\to a} |f(x)| = 0$ if and only if $\lim_{x\to a} f(x) = 0$.

7. Prove Squeeze Theorem for Functions.

8. Prove Comparision Theorem for Functions.

9. Suppose that f is a real function.

   9.1 Prove that if

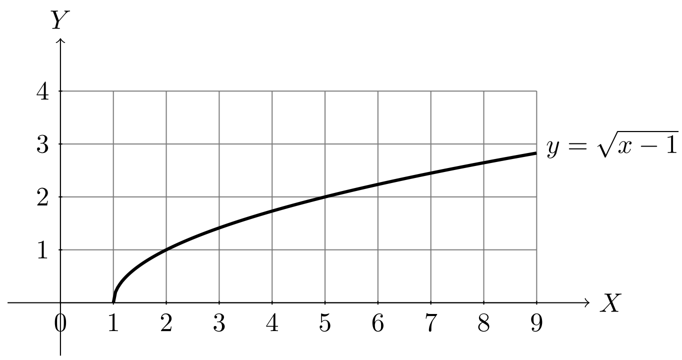
   $$\lim_{x\to a} f(x) = L$$

   exists, then $|f(x)| \to |L|$ as $x \to a$.

   9.2 Show that there is a function such that as $x \to a$, $|f(x)| \to |L|$ but the limit of $f(x)$ does not exist.

4.2 One-sided limit

What is the limit of $f(x) := \sqrt{x-1}$ as $x \to 1$.



A reasonable answer is that the limit is zero. This function, however, does not satisfy Definition 4.1.1 because it is not defined on an OPEN interval containg $a = 1$. Indeed, f is defined only for $x \geq 1$. To handle such situations, we introduce one-sided limits.

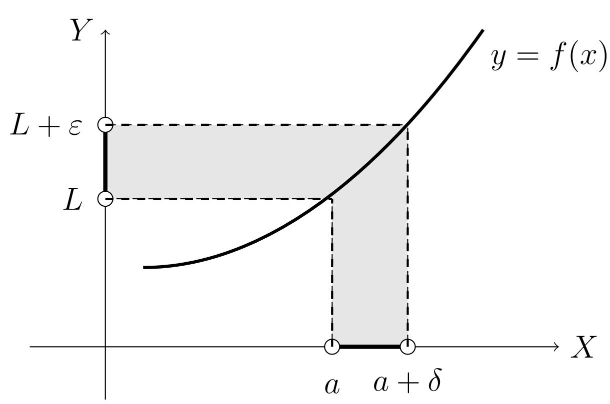
Definition 4.2.1 *Let* $a \in \mathbb{R}$.

1. *A real function* f *said to* ***converge*** *to* L *as* x ***approaches*** a ***from the right*** *if and only if* f *defined on some interval* I *with left endpoint* a *and every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $a + \delta \in I$ *and for all* $x \in I$,

$$a < x < a + \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we call L *the* ***right-hand limit*** *of* f *at* a, *and denote it by*

$$f(a^+) := L =: \lim_{x \to a^+} f(x).$$

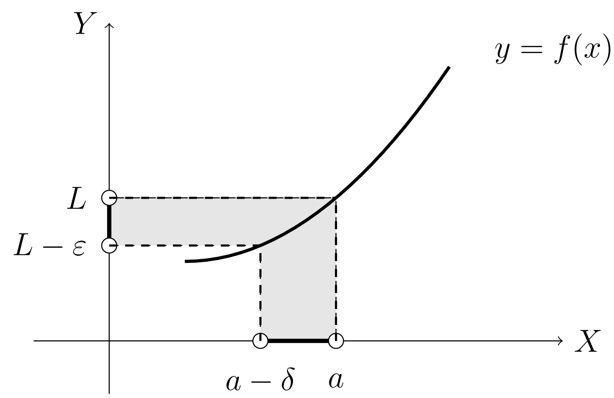


2. *A real function f said to* ***converge*** *to L as x* ***approaches*** *a* ***from the left*** *if and only if f defined on some interval I with right endpoint a and every $\varepsilon > 0$ there is a $\delta > 0$ such that $a + \delta \in I$ and for all $x \in I$,*

$$a - \delta < x < a \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we call L the ***left-hand limit*** *of f at a, and denote it by*

$$f(a^-) := L =: \lim_{x \to a^-} f(x).$$



Example 4.2.2 *Prove that* $\lim_{x \to 1^+} \sqrt{x-1} = 0.$

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2 > 0$. Let $x > 1$ such that $0 < x - 1 < \delta$. We obtain

$$|f(x) - 0| = |\sqrt{x-1} - 0| = \sqrt{x-1} < \sqrt{\delta} = \varepsilon.$$

Thus, $\sqrt{x-1} \to 0$ as $x \to 1^+$. □

Example 4.2.3 *If* $f(x) = \dfrac{|x|}{x}$, *prove that f has one-sided limit at $a = 0$ but* $\lim_{x \to 0} f(x) = 0$ *DNE.*

Solution. Let $\varepsilon > 0$. We can choose any $\delta > 0$. Let $x \in \mathbb{R} \backslash \{0\}$ such that $-\delta < x < 0$. Then $|x| = -x$. We obtain

$$|f(x) - 0| = \left| \frac{|x|}{x} - (-1) \right| = \left| \frac{-x}{x} - (-1) \right| = |-1 + 1| = 0 < \varepsilon.$$

Thus, $\lim_{x \to 0^-} f(x) = -1$. Similarly, $\lim_{x \to 0^-} f(x)$ exists and equals 1.
Choose two sequence as follow

$$\begin{array}{rclcccrcl} x_n & = & \dfrac{1}{n} & \to 0 & \text{and} & f(x_n) & = & 1 & \to & 1, \\ y_n & = & -\dfrac{1}{n} & \to 0 & \text{and} & f(y_n) & = & -1 & \to & -1. \end{array}$$

Then $f(x_n)$ and $f(y_n)$ converge to distinct limits. By SCL, we conclude that f has no limit as $x \to 0$.

Theorem 4.2.4 *Let f be a real function. Then the limit*

$$\lim_{x \to a} f(x)$$

exists and equals to L if and only if

$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Proof. Assume that $f(x) \to L$ as $x \to a$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{and} \quad |f(x) - L| < \varepsilon. \tag{4.3}$$

If $a < x < a + \delta$, it satisfies (4.3) which implies $|f(x) - L| < \varepsilon$. Thus, $\lim_{x \to a^+} f(x) = L$.
If $a - \delta < x < a$, it satisfies (4.3) which implies $|f(x) - L| < \varepsilon$. Thus, $\lim_{x \to a^-} f(x) = L$.

Conversely, assume that $L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$. Let $\varepsilon > 0$. There are $\delta_1, \delta_2 > 0$ such that

$$a < x < a + \delta_1 \quad \to \quad |f(x) - L| < \varepsilon \tag{4.4}$$

and

$$a - \delta_2 < x < a \quad \to \quad |f(x) - L| < \varepsilon. \tag{4.5}$$

Choose $\delta = \min\{\delta_1, \delta_1\}$. If $|x - a| < \delta$, it satisfies (4.4) and (4.5) which imply

$$|f(x) - L| < \varepsilon.$$

Therefore, $\lim_{x \to a} f(x) = L$. □

Example 4.2.5 *Use Theorem 4.2.4 to show that* $f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$ *has limit at $a = 0$.*

Solution. We see that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1 = \lim_{x \to 0^-} (2x + 1) = \lim_{x \to 0^-} f(x).$$

By Theorem 4.2.4, we conclude that $\lim_{x \to 0} f(x) = 1$

Exercises 4.2

1. Use definitons to prove that $\lim\limits_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = 2x^2 + 1,$ $\qquad a = 1$, and $L = 3$.

   1.2 $f(x) = \dfrac{x-1}{|1-x|},$ $\qquad a = 1$, and $L = 1$.

   1.3 $f(x) = \sqrt{3x-5},$ $\qquad a = 2$, and $L = 1$.

2. Use definitons to rove that $\lim\limits_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = 1 + x^2,$ $\qquad a = 1$, and $L = 2$.

   2.2 $f(x) = \sqrt{1-x^2},$ $\qquad a = 1$, and $L = 0$.

   2.3 $f(x) = \dfrac{1-x^2}{1+x},$ $\qquad a = 1$, and $L = 0$.

3. Evauate the following limit when they exist.

   3.1 $\lim\limits_{x\to 0^+} \dfrac{x+1}{x^2-2}$

   3.2 $\lim\limits_{x\to 1^-} \dfrac{x^3-3x+2}{x^3-1}$

   3.3 $\lim\limits_{x\to \pi^+} (x^2+1)\sin x$

   3.4 $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{\cos x}{1-\sin x}$

4. Prove that $\dfrac{\sqrt{1-\cos x}}{\sin x} \to \dfrac{\sqrt{2}}{2}$ as $x \to 0^+$.

5. Determine whether the following functions are limit at a.

   5.1 $f(x) = \begin{cases} 3x+1 & \text{if } x \geq 1 \\ x+3 & \text{if } x < 1 \end{cases}$ and $\quad a = 1$

   5.2 $f(x) = \begin{cases} 2-2x & \text{if } x \geq 0 \\ \sqrt{1-x} & \text{if } x < 0 \end{cases}$ and $\quad a = 0$

6. Suppose that $f : [0,1] \to \mathbb{R}$ and $f(a) = \lim\limits_{x\to a} f(x)$ for all $x \in [0,1]$. Prove that

$$f(q) = 0 \text{ for all } q \in \mathbb{Q} \cap [0,1] \text{ if and only if } f(x) = 0 \text{ for all } x \in [0,1].$$

4.3 Infinite limit

The definition of limit of real functions can be expanded to include extended real numbers.

Definition 4.3.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to L$ as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x > M \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we shall write $\lim_{x\to\infty} f(x) = L$.

2. *We say that $f(x) \to L$ as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(-\infty, -c) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x < M \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we shall write $\lim_{x\to-\infty} f(x) = L$.

Example 4.3.2 *Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.*

Proof. Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon} > 0$. If $x > M > 0$, it implies

$$\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \varepsilon.$$

We conclude that $\lim_{x\to\infty} \frac{1}{x} = 0$. □

Example 4.3.3 *Prove that $\lim_{x\to\infty} \frac{x-1}{x+1}$ exists and equals to 1.*

Proof. Let $\varepsilon > 0$. Choose $M = \frac{2}{\varepsilon} > 0$. If $x > M > 0$, it follows that $x + 1 > x > M$. So, $\frac{1}{x+1} < \frac{1}{M}$. We obtain

$$\left|\frac{x-1}{x+1} - 1\right| = \left|\frac{-2}{x+1}\right| = 2 \cdot \frac{1}{x+1} < \frac{2}{M} = \varepsilon.$$

We conclude that $\lim_{x\to\infty} \frac{x-1}{x+1} = 1$. □

Example 4.3.4 *Prove that* $\lim_{x\to\infty} \frac{1}{x^2+1} = 0.$

Proof. Let $\varepsilon > 0$. Choose $M = \frac{1}{\sqrt{\varepsilon}} > 0$. If $x > M > 0$, it follows that $x^2 > M^2 > 0$. So, $\frac{1}{x^2} < \frac{1}{M^2}$. We obtain

$$\left|\frac{1}{x^2+1} - 0\right| = \frac{1}{x^2+1} < \frac{1}{x^2} < \frac{1}{M^2} = \varepsilon.$$

We conclude that $\lim_{x\to\infty} \frac{1}{x^2+1} = 0.$ □

Example 4.3.5 *Prove that* $\lim_{x\to-\infty} \frac{1}{x} = 0.$

Proof. Let $\varepsilon > 0$. Choose $M = -\frac{1}{\varepsilon} < 0$. If $x < M < 0$, it implies $-x > -M > 0$. We obtain

$$\left|\frac{1}{x} - 0\right| = \frac{1}{-x} < \frac{1}{-M} = \varepsilon.$$

We conclude that $\lim_{x\to-\infty} \frac{1}{x} = 0.$ □

Example 4.3.6 *Prove that* $\lim_{x\to-\infty} \frac{x}{x+1} = 1.$

Proof. Let $\varepsilon > 0$. Choose $M = -1 - \frac{1}{\varepsilon}$. Then $M + 1 = -\frac{1}{\varepsilon} < 0$. If $x < M$, it implies $1 + x < 1 + M < 0$. So, $0 < -\frac{1}{x+1} < -\frac{1}{M+1}$. We obtain

$$\left|\frac{x}{x+1} - 1\right| = \frac{1}{|x+1|} = \frac{1}{-(x+1)} < \frac{1}{-(M+1)} = \varepsilon.$$

We conclude that $\lim_{x\to-\infty} \frac{x}{x+1} = 1.$ □

Definition 4.3.7 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to +\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M > 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \textit{implies} \quad f(x) > M.$$

   In this case we shall write $\lim_{x \to a} f(x) = +\infty$.

2. *We say that $f(x) \to -\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M < 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \textit{implies} \quad f(x) < M.$$

   In this case we shall write $\lim_{x \to a} f(x) = -\infty$.

Obviousl modification define $f(x) \to \pm\infty$ as $x \to a^+$ and $x \to a^-$, and $f(x) \to \pm\infty$ as $x \to \pm\infty$.

Example 4.3.8 *Prove that* $\lim_{x \to 0} \dfrac{1}{|x|} = +\infty$.

Proof. Let $M > 0$. Choose $\delta = \dfrac{1}{M} > 0$. If $0 < |x| < \delta$, it follows

$$\frac{1}{|x|} > \frac{1}{\delta} = M.$$

Thus, $\lim_{x \to 0} \dfrac{1}{|x|} = +\infty$. □

Example 4.3.9 *Prove that* $\lim_{x \to 1^+} \dfrac{x}{1 - x} = -\infty$.

Proof. Let $M < 0$. Choose $\delta = -\dfrac{1}{M} > 0$. If $0 < x - 1 < \delta$, it follows $\dfrac{1}{\delta} < \dfrac{1}{x - 1}$. So, $\dfrac{1}{1 - x} < -\dfrac{1}{\delta}$. We obatin

$$\frac{x}{1 - x} = -1 + \frac{1}{1 - x} < 0 + \frac{1}{1 - x} < -\frac{1}{\delta} = M.$$

Thus, $\lim_{x \to 1^+} \dfrac{x}{1 - x} = -\infty$. □

Exercises 4.3

---

1. Use definitons to prove that $\lim\limits_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = \dfrac{1}{x-3}$, $\quad a = 3$, and $L = +\infty$.

   1.2 $f(x) = -\dfrac{1}{x}$, $\quad a = 0$, and $L = -\infty$.

2. Use definitons to prove that $\lim\limits_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = \dfrac{x}{x^2-4}$, $\quad a = 2$, and $L = -\infty$.

   2.2 $f(x) = \dfrac{1}{1-x^2}$, $\quad a = 1$, and $L = +\infty$.

3. Use definition to prove that the follwing limits

   3.1 $\lim\limits_{x\to\infty} \dfrac{2x+1}{x+1} = 2$

   3.2 $\lim\limits_{x\to-\infty} \dfrac{1-x}{2x+1} = -\dfrac{1}{2}$

   3.3 $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{1-x^2} = -2$

   3.4 $\lim\limits_{x\to 2} \dfrac{x}{|x-2|} = +\infty$

   3.5 $\lim\limits_{x\to 2^+} \dfrac{x+1}{x-2} = +\infty$

   3.6 $\lim\limits_{x\to 2^-} \dfrac{x+1}{x-2} = -\infty$

4. Evauate the following limit when they exist.

   4.1 $\lim\limits_{x\to\infty} \dfrac{3x^2-13x+4}{1-x-x^2}$

   4.2 $\lim\limits_{x\to\infty} \dfrac{x^2+x+2}{x^3-x-2}$

   4.3 $\lim\limits_{x\to-\infty} \dfrac{x^3-1}{x^2+2}$

   4.4 $\lim\limits_{x\to\infty} \arctan x$

   4.5 $\lim\limits_{x\to\infty} \dfrac{\sin x}{x^2}$

   4.6 $\lim\limits_{x\to-\infty} x^2 \sin x$

5. Prove that $\dfrac{\sin(x+3)-\sin 3}{x}$ converges to 0 as $x \to \infty$.

6. Prove the following comparision theorems for real functions.

   6.1 If $f(x) \geq g(x)$ and $g(x) \to \infty$ as $x \to a$, then $f(x) \to \infty$ as $x \to a$.

   6.2 If $f(x) \leq g(x) \leq h(x)$ and $L = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} h(x)$, then $g(x) \to L$ as $x \to \infty$.

7. Recall that a **polynomial of degree n** is a functon of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_j \in \mathbb{R}$ for $j = 0, 1, ..., n$ and $a_n \neq 0$.

7.1 Prove that $\lim\limits_{x \to a} x^n = a^n$ for $n = 0, 1, 2, ...$

7.2 Prove that if P is a polynomial, then

$$\lim_{x \to a} P(x) = P(a)$$

for every $a \in \mathbb{R}$.

7.3 Suppose that P is a polynomial and $P(a) > 0$. Prove that $\dfrac{P(x)}{x-a} \to \infty$ as $x \to a^+$, $\dfrac{P(x)}{x-a} \to -\infty$ as $x \to a^-$, but

$$\lim_{x \to a} \frac{P(x)}{x-a}$$

does not exist.

8. **Cauchy**. Suppose that $f : \mathbb{N} \to \mathbb{R}$. If

$$\lim_{n \to \infty} f(n+1) - f(n) = L,$$

prove that $\lim\limits_{n \to \infty} \dfrac{f(n)}{n}$ exists and equals L.

Chapter 5

Continuity on $\mathbb{R}$

5.1 Continuity

Definition 5.1.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*
f is said to be ***continuous*** *at a point $a \in E$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \text{ and } x \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.1.2 *Let $f(x) = 2x - 1$ where $x \in \mathbb{R}$. Prove that f is continuous at $x = 1$.*

Proof. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2} > 0$. Let $x \in \mathbb{R}$ such that $|x - 1| < \delta$. We obtain

$$|f(x) - f(1)| = |(2x - 1) - 1| = |2(x - 1)| = 2|x - 1| < 2\delta = \varepsilon.$$

Thus, f is continuous at $x = 1$. □

Example 5.1.3 *Let $f(x) = x^2$ where $x \in \mathbb{R}$. Prove that f is continuous at $x = 2$.*

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \dfrac{\varepsilon}{5}\right\}$. Let $x \in \mathbb{R}$ such that $|x - 2| < \delta$.
We obtain $|x| - 2 < |x - 2| < 1$. It follows $|x| < 3$. So,

$$|f(x) - f(2)| = |x^2 - 4| = |x + 2||x - 2| < (|x| + 2)\delta < (3 + 2)\frac{\varepsilon}{5} = \varepsilon.$$

Thus, f is continuous at $x = 2$. □

Example 5.1.4 *Let $f(x) = \sqrt{x}$ where $x \in (0, \infty)$. Prove that f is continuous at 1.*

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Let $x \in (0, \infty)$ such that $|x - 1| < \delta$.
Since $\sqrt{x} + 1 > 1$, $\dfrac{1}{\sqrt{x}+1} < 1$. We obtain

$$\begin{aligned} |f(x) - f(1)| &= |\sqrt{x} - 1| \\ &= \left|(\sqrt{x} - 1) \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}\right| = \left|\frac{x-1}{\sqrt{x}+1}\right| = |x - 1| \cdot \frac{1}{\sqrt{x}+1} < \delta \cdot 1 = \varepsilon. \end{aligned}$$

Thus, f is continuous at $x = 2$. □

Example 5.1.5 *Let $f(x) = 3 - x^2$ where $x \in [-1, 2] \cup \{3\}$. Prove that f is continuous at $x = 3$*

Proof. Let $\varepsilon > 0$. Choose $\delta = 0.5$. Let $x \in [-1, 2] \cup \{3\}$ such that $|x - 3| < \delta = 0.5$. It follows $x = 3$. We obtain

$$|f(x) - f(3)| = |f(3) - f(3)| = 0 < \varepsilon.$$

Thus, f is continuous at $x = 3$. □

Example 5.1.6 *Prove that the function*

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0.

Proof. Suppose that f is continuous at 0. Given $\varepsilon = 1$. There is a $\delta > 0$ such that

$$|x| < \delta \text{ and } x \in \mathbb{R} \quad \text{imply} \quad |f(x)| = |f(x) - f(0)| < 1. \tag{5.1}$$

For $0 < x < \delta$, we obtain by (5.1) such that

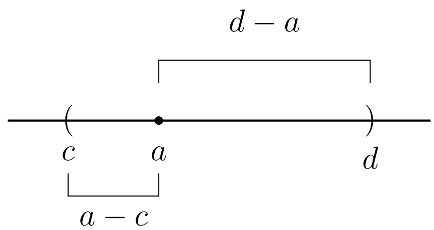
$$1 = \frac{x}{x} = \left|\frac{|x|}{x}\right| = |f(x)| < 1$$

It is imposible. Thus, f is discontinuous at 0. □

Theorem 5.1.7 *Let I be an open interval that contain a point a and $f : I \to \mathbb{R}$. Then*

$$f \text{ is continuous at } a \in I \text{ if and only if } \quad f(a) = \lim_{x \to a} f(x).$$

Proof. Let $I = (c, d)$ such that contain a point a.



Set $\delta_0 = \min\{a - c, d - a\}$. Choose $\delta < \delta_0$. Then $|x - a| < \delta$ implies $x \in I$.

Therefore, conditions

$$|x - a| < \delta \text{ and } x \in I \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon$$

is identical to

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - f(a)| < \varepsilon.$$

We conclude that f is continuous at $a \in I$ if and only if $f(a) = \lim_{x \to a} f(x)$. □

Example 5.1.8 *Let $f(x) = x \cos\left(\frac{1}{x}\right)$ where $x \neq 0$. If f is continuous at 0, what is $f(0)$ defined?*

Solution. Use Example 4.1.18 and Theorem 5.1.7 in order to define

$$f(0) = \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$$

Thus, we define $f(0) = 0$ that makes f be continuous at 0.

Example 5.1.9 *Find a such that the function $f(x) = \begin{cases} ax + 1 & \text{if } x \geq 1 \\ 2x + 3 & \text{if } x < 1 \end{cases}$ is continuous at 1.*

Solution. From f is continuous at 1, we obtain

$$\begin{aligned} f(1) &= \lim_{x \to 1^+} f(x) &&= \lim_{x \to 1^-} f(x) \\ a + 1 &= \lim_{x \to 1^+} (ax + 1) &&= \lim_{x \to 1^-} (2x + 3) \\ a + 1 &= a + 1 &&= 5 \end{aligned}$$

Hence, $a = 4$.

Theorem 5.1.10 *Suppose that E is a nonempty subset of $\mathbb{R}$, $a \in E$, and $f : E \to \mathbb{R}$. Then the following statements are equivalent:*

1. *f is continuous at $a \in E$.*
2. *If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.*

Proof. The proof Theorem is complete by Theorem 5.1.7 and SCL. □

Example 5.1.11 *Use Theorem 5.1.10 to find* $\lim_{n\to\infty} \sqrt{\dfrac{n}{n+1}}$.

Solution. Let $f(x) = \sqrt{x}$ where $x \in (0, \infty)$. By Example 5.1.4, f is continuos at 1. Set

$$x_n = \frac{n}{n+1}.$$

Then $\lim_{n\to\infty} x_n = 1$ by Example 2.1.6. By Theorem 5.1.7, it implies that

$$f(x_n) = \sqrt{\frac{n}{n+1}} \to f(1) = 1.$$

Next, we will use Theorem 5.1.10 together Theorems of limit for addition, mutiplication, scalar multiplication and quotient in order to proof Theorem 5.1.12.

Theorem 5.1.12 *Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. If f, g are continuous at a point $a \in E$, then so are*

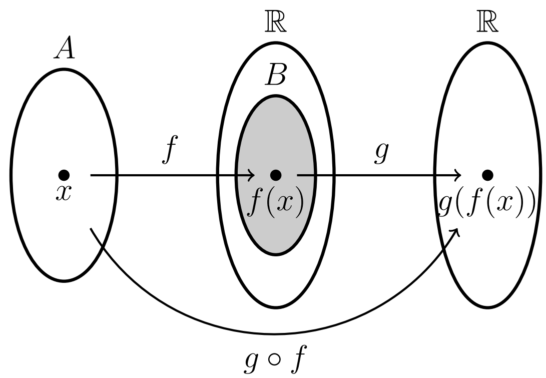
$$f + g, \quad fg \quad \text{and} \quad \alpha f$$

Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$.

CONTINUITY OF COMPOSITION.

Definition 5.1.13 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $\{f(x) : x \in A\} \subseteq B$, then the* ***composition*** *of g with f is the function*

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$



Theorem 5.1.14 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $\{f(x) : x \in A\} \subseteq B$. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then*

$$g \circ f \text{ is continuous at } a \in A$$

and moreover,

$$\lim_{x\to a}(g \circ f)(x) = g\left(\lim_{x\to a} f(x)\right).$$

Proof. Assume that f is continuous at $a \in A$ and g is continuous at $f(a) \in B$.
Let $\varepsilon > 0$. There is a $\delta_1 > 0$ such that

$$|y - f(a)| < \delta_1 \text{ and } y \in B \quad \text{imply} \quad |g(y) - g(f(a))| < \varepsilon. \tag{5.2}$$

There is a $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \text{ and } x \in A \quad \text{imply} \quad |f(x) - f(a)| < \delta_1. \tag{5.3}$$

For each $x \in A$ such that $|x - a| < \delta_2$, it implies $|f(x) - f(a)| < \delta_1$. Set $y = f(x)$.
We obtain by (5.2) that $|g(f(x)) - g(f(a))| < \varepsilon$. We conclude that $g \circ f$ is continuous at $a \in A$. □

Example 5.1.15 *Show that* $\lim_{x\to 1}\sqrt{2x-1}$ *exists and equals to 1.*

Solution. Let $g(x)=\sqrt{x}$ and $f(x)=2x-1$. Then f is continous at 1 and g is continuous at $f(1)=1$. By Theorem 5.1.14,

$$\lim_{x\to 1}(g\circ f)(x) = g\left(\lim_{x\to 1} f(x)\right) = g\left(\lim_{x\to 1}(2x-1)\right) = g(1) = 1.$$

CONTINUITY ON A SET.

Definition 5.1.16 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*

f is said to be ***continuous on E*** *if and only if f is continuous at every $a \in E$.*

Note that if f is continuous on E, then f is continuous on nonempty subset of E.

Example 5.1.17 *Show that $f(x)=x^2$ is continuous on $\mathbb{R}$.*

Proof. Let $a\in\mathbb{R}$ and $\varepsilon>0$. Choose $\delta=\min\left\{1,\dfrac{\varepsilon}{2|a|+1}\right\}$.
Let $x\in\mathbb{R}$ such that $|x-a|<\delta$. We obtain $|x|-|a|<|x-a|<1$. It follows

$$|x| < 1+|a|.$$

We obtain

$$\begin{aligned}|f(x)-f(a)| &= |x^2-a^2| = |x+a||x-a|\\ &< (|x|+|a|)\delta < (|a|+1+|a|)\frac{\varepsilon}{2|a|+1} = \varepsilon.\end{aligned}$$

Thus, f is continuous on $\mathbb{R}$. □

Theorem 5.1.18 (**Continuity of linear function**) *Let m and c be constants and let*

$$f(x) = mx + c \text{ where } x \in \mathbb{R}.$$

Prove that f is continuous on $\mathbb{R}$

Proof. Let $a\in\mathbb{R}$ and $\varepsilon>0$. Choose $\delta=\dfrac{\varepsilon}{|m|+1}>0$. Let $x\in\mathbb{R}$ such that $|x-a|<\delta$. We obtain

$$\begin{aligned}|f(x)-f(a)| &= |(mx+c)-(ma+c)| = |m||x-a|\\ &< |m|\delta \le |m|\cdot\frac{\varepsilon}{|m|+1} < 1\cdot\varepsilon = \varepsilon.\end{aligned}$$

Thus, f is continuous at $\mathbb{R}$. □

Example 5.1.19 *Show that $h(x) = (3x+1)^2$ is continuous on $\mathbb{R}$.*

Solution. Let $f(x) = x^2$ and $g(x) = 3x + 1$. By Example 5.1.17 and Theorem 5.1.18, f and g are continuous on $\mathbb{R}$. We conclude by Theorem 5.1.14 that

$$h(x) = f \circ g(x) = (3x+1)^2 \text{ is continuous on } \mathbb{R}.$$

Example 5.1.20 *Prove that*

$$f(x) = \begin{cases} 2x+4 & \text{if } x > -1 \\ 3x+5 & \text{if } x \leq -1 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution. We see that f is a linear function on $(-1, \infty) \cup (-1, \infty)$. By Continuity of Linear function, f is continuous on $(-1, \infty) \cup (-1, \infty)$. From

$$f(-1) = 2 = \lim_{x \to -1^+} (3x+5) = \lim_{x \to -1^-} (2x+4),$$

it follows that f is continuous at -1. We conclude that f is is continuous on $\mathbb{R}$.

Example 5.1.21 *Find a such that the function $f(x) = \begin{cases} ax+1 & \text{if } x \geq 2 \\ x+a & \text{if } x < 2 \end{cases}$ is continuous on $\mathbb{R}$.*

Solution. From f is continuous at 2, we obtain

$$\begin{aligned} f(2) &= \lim_{x \to 2^+} f(x) &&= \lim_{x \to 2^-} f(x) \\ 2a+1 &= \lim_{x \to 2^+} (ax+1) &&= \lim_{x \to 2^-} (x+a) \\ 2a+1 &= 2a+1 &&= 2+a. \end{aligned}$$

Hence, $a = 1$.

Exercises 5.1

1. Use definition to prove that f is continuous at a.

   1.1 $f(x) = x^2 + 1$ and $a = 1$.

   1.2 $f(x) = x^3$ and $a = -1$.

   1.3 $f(x) = \dfrac{1}{x}$ and $a = 1$.

   1.4 $f(x) = \dfrac{x}{x^2 + 1}$ and $a = 2$.

2. Determine whether the following functions are continuous at a.

   2.1 $f(x) = \begin{cases} 1 - 2x & \text{if } x \geq 1 \\ 2 - 3x & \text{if } x < 1 \end{cases}$ and $a = 1$

   2.2 $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 0 \\ \sqrt{1 - x} & \text{if } x < 0 \end{cases}$ and $a = 0$

3. Use definition to prove that f is continuous at E.

   3.1 $f(x) = x^3$ and $E = \mathbb{R}$.

   3.2 $f(x) = \sqrt{1 - x}$ and $E = (-\infty, 1)$.

   3.3 $f(x) = \dfrac{1}{x^2 + 1}$ and $E = \mathbb{R}$.

4. Use limit theorem to show that the following function are continuous on $[0, 1]$.

   4.1 $f(x) = 3x^2 + 1$

   4.2 $f(x) = \dfrac{1 - x}{1 + x}$

   4.3 $f(x) = \sqrt{2 - x}$

   4.4 $f(x) = \dfrac{1}{x^2 + x - 6}$

5. Find a and b such that the function $f(x) = \begin{cases} ax + 3 & \text{if } x \leq 1 \\ x + b & \text{if } 1 < x \leq 2 \\ 2ax - 2 & \text{if } x > 2 \end{cases}$ is continuous on $\mathbb{R}$.

6. If $f : [a, b] \to \mathbb{R}$ is continuous, prove that $\sup\limits_{x \in [a,b]} |f(x)|$ is finite.

7. Show that there exist nowhere continuous functions f and g whose sum $f + g$ is continuous on $\mathbb{R}$. Show that the same is ture for product of functions.

8. Let

$$f(x) = \begin{cases} \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, 0)$ and $(0, \infty)$, discontinuous at 0, and neither $f(0^+)$ nor $f(0^-)$ exists.

   8.1 Prove that f is continuous on $(-\infty, 0)$ and $(0, \infty)$ discontinuous at 0.

   8.2 Suppose that $g : [0, \frac{2}{\pi}] \to \mathbb{R}$ is continuous on $(0, \frac{2}{\pi})$ and that there is a positive constant $C > 0$ such that

   $$|g(x)| \leq C\sqrt{x} \text{ for all } x \in (0, \tfrac{2}{\pi}),$$

   Prove that $f(x)g(x)$ is continuous on $[0, \frac{2}{\pi}]$.

9. Suppose that $a \in \mathbb{R}$, that I is an open interval containing a, that, $f, g : I \to \mathbb{R}$, and that f is continuous at a.

   9.1 Prove that g is continuous at a if and only if $f + g$ is continuous at a.

   9.2 Make and prove an analogous atstement for the product fg. Show by example that hypothesis about f added cannot be dropped.

10. Let $f : A \to \mathbb{R}$ be a continuous function. Suppose that $E \subseteq A$ and is open. Determine whether $\{f(x) : x \in E\}$ is open.

11. Let $f(x) = x^n$ where $n \in \mathbb{N}$. Prove that f is continuous on $\mathbb{R}$

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

   12.1 Show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

   12.2 Prove that $f(qx) = qf(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q}$.

   12.3 Prove that f is continuous at 0 if and only if f is continuous on $\mathbb{R}$.

   12.4 Prove that f is continuous at 0, then there is an $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

13. Assume that $\lim_{n\to 0} \dfrac{\ln(x+1)}{x} = 1$ and $f(x) = e^x$ is continuous on $\mathbb{R}$. Show that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

5.2 Intermediate Value Theorem

Definition 5.2.1 *Let E be a nonempty subsets of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be* ***bounded on E*** *if and only if there is an $M > 0$ such that*

$$|f(x)| \leq M \quad \text{for all } x \in E$$

For a example $f(x) = \sin x$, by sine property that

$$|\sin x| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

So, f is bounded by 1 on $\mathbb{R}$.

Next, let $f : I \to \mathbb{R}$ be a function. We define

$$\sup_{x\in I} f(x) := \sup\{f(x) : x \in I\}$$

$$\inf_{x\in I} f(x) := \inf\{f(x) : x \in I\}$$

For example $\sup\limits_{x\in[0,1)} x^2 = 1$ and $\inf\limits_{x\in[0,1)} x^2 = 0$.

Theorem 5.2.2 (**Extreme Value Theorem (EVT)**) *If I is a closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover, if*

$$M = \sup_{x\in I} f(x) \quad \text{and} \quad m = \inf_{x\in I} f(x),$$

then there exist point $x_m, x_M \in I$ such that

$$f(x_M) = M \quad \text{and} \quad f(x_m) = m.$$

Proof. Suppose that f is not bounded in I. Then there exist $x_n \in I$ such that

$$|f(x_n)| > n \quad \text{for } n \in \mathbb{N} \tag{5.4}$$

Since I is bounded, we know by the Bolzano-Weierstrass Theorem that $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Since I is closed, we also know by the Comparison Theorem that $a \in I$ and $f(a) \in \mathbb{R}$. By (5.4), we obtain

$$f(a) = \lim_{k\to\infty} |f(x_{n_k})| > \lim_{k\to\infty} n_k \geq \lim_{k\to\infty} k = \infty$$

which contradics $f(a) \in \mathbb{R}$. Thus, f is bounded in I.

We will prove that M and m are finite real numbers. Suppose that

$$f(x) < M = \sup_{x \in I} f(x) \quad \text{for all } x \in I.$$

Then the function

$$g(x) = \frac{1}{M - f(x)} \text{ is continuous on } I.$$

So, g is bounded on I. There i a $C > 0$ such that $|g(x)| = g(x) \leq C$ for all $x \in I$. It follows that

$$f(x) \leq M - \frac{1}{C}.$$

We obtain

$$M = \sup_{x \in I} f(x) \leq M - \frac{1}{C} < M.$$

It is imposible. Thus, there is an $x_M \in I$ such that $f(x_M) = M$. A similar argument proves that there is an $x_m \in I$ such that $f(x_m) = m$. □

Lemma 5.2.3 (**Sign-Preserving Property**) *Let $f : I \to \mathbb{R}$ where I is open. If f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$, then there are positive numbers ε and δ such that*

$$|x - x_0| < \delta \quad \textit{implies} \quad f(x) > \varepsilon.$$

Proof. Assume that f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$.
Given $\varepsilon = \dfrac{f(x_0)}{2}$. There is a $\delta > 0$ such that

$$|x - x_0| < \delta \text{ and } x \in I \quad \text{imply} \quad |f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

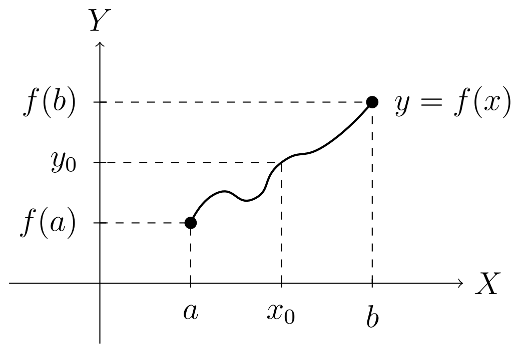
It follows that

$$\begin{array}{ccccc} -\dfrac{f(x_0)}{2} & < & f(x) - f(x_0) & < & \dfrac{f(x_0)}{2} \\ \dfrac{f(x_0)}{2} & < & f(x) & < & \dfrac{3f(x_0)}{2} \end{array}$$

Thus, $f(x) > \dfrac{f(x_0)}{2} = \varepsilon$. □

Theorem 5.2.4 (**Intermediate Value Theorem (IVT)**) *Let $f : [a, b] \to \mathbb{R}$ be continuous. If y_0 lies between $f(a)$ and $f(b)$, then*

$$\textit{there is an } x_0 \in (a, b) \textit{ such that } f(x_0) = y_0.$$



Proof. We may suppose that $f(a) < y_0 < f(b)$. Consider

$$E = \{x \in [a, b] : f(x) < y_0\}.$$

Since $a \in E$ and $E \subseteq [a, b]$, E is a nonempty bounded subset of $\mathbb{R}$. Thus, by the Completeness Axiom, $x_0 = \sup E$ is a finite real number. Since y_0 is equals neither $f(a)$ nor $f(b)$, x_0 cannot equal to a or b. Hence, $x_0 \in (a, b)$.

It remains to show that $f(x_0) = y_0$. By Theorem 2.2.5, there is a sequence $x_n \in E$ such that

$$x_n \to \sup E = x_0 \text{ as } n \to \infty.$$

Since f is continuous and the definition of E, by the Comparison Theorem and Theorem 5.1.10 we obtain

$$f(x_0) = \lim_{n\to\infty} f(x_n) \leq y_0.$$

Finally, we will prove that $f(x_0) = y_0$, suppose to the contrary that $f(x_0) < y_0$. Set

$$g(x) = y_0 - f(x) \qquad \text{where } x \in E.$$

Then g is continuous and $g(x_0) > 0$. Hence, by Lemma 5.2.3, we can choose positive numbers ε and δ such that

$$|x - x_0| < \delta \qquad \text{implies} \qquad g(x) > \varepsilon > 0.$$

For any x, it satisfies $x_0 < x < x_0 + \delta$ also satisfies $y_0 - f(x) = g(x) > 0$ or $f(x) < y_0$ which contradics the fact that $x_0 = \sup E$. $\square$

Corollary 5.2.5 *Let $f : [a, b] \to \mathbb{R}$ be continuous.*

1. *If $f(a) > 0$ and $f(b) < 0$, then there is an $c \in (a, b)$ such that $f(c) = 0$.*

2. *If $f(a) < 0$ and $f(b) > 0$, then there is an $c \in (a, b)$ such that $f(c) = 0$.*

Proof. It is obviously by the IVT. □

Example 5.2.6 *Show that there is a real number such that $x^2 = x + 1$.*

Solution. Let $f(x) = x^2 - x - 1$. Then $f(1) = -1 < 0$ and $f(2) = 2 > 0$.
Since f is continuous on $(1, 2)$, we obatin by Corollary 5.2.5 that there is an $c \in (1, 2)$ such that

$$c^2 - c - 1 = f(c) = 0.$$

Thus, there exists a real number c such that $c^2 = c + 1$.

Example 5.2.7 *Prove that $\ln x = 3 - 2x$ has at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ of length 0.01 that contain a root.*

Solution. Let $f(x) = \ln x + 2x - 3$. Consider each values of $f(x)$ by calculator

x	$f(x)$	Interval	Length of Interval
2	1.6931		
1	-1	$[1, 2]$	1
1.4	0.1365		
1.3	-0.1376	$[1.3, 1.4]$	0.1
1.35	0.00010		
1.34	-0.02733	$[1.34, 1.35]$	0.01

Since f is continuous on $(1.34, 1.35)$, we obatin by Corollary 5.2.5 that there is an $c \in (1.34, 1.35)$ such that

$$\ln c + 2c - 3 = f(c) = 0.$$

Thus, there exists a real number c such that $\ln c = 3 - 2c$.

We may approximate the root by choosing midpoint $c = 1.345$ of $(1.34, 1.35)$. It follows that $f(c) = -0.0136$ which has error 0.01.

Exercises 5.2

For these exercise, assume that $\sin x$, $\cos x$ and e^x are continuous on $\mathbb{R}$ and $\ln x$ is continuous on $\mathbb{R}^+$.

1. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the given equation.

1.1 $x^3 + x = 3$	1.6 $e^x = x^2$
1.2 $x^3 + 2 = 2x$	1.7 $x \ln x = 1$
1.3 $x^4 + x^3 - 2 = 0$	1.8 $\sin x = e^x$
1.4 $x^5 + x + 1 = 0$	1.9 $\cos x = x^2$
1.5 $2^x = 2 - x$	1.10 $e^x = \cos x + 1$

2. Prove that the follwing equations have at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ of length ℓ that contain a root.

2.1 $x^3 + x = 1$	and $\ell = 0.001$	2.4 $\cos x = x$	and $\ell = 0.01$
2.2 $2^x = x^3$	and $\ell = 0.01$	2.5 $\sin x + x = 1$	and $\ell = 0.001$
2.3 $\ln x + x = 2$	and $\ell = 0.001$	2.6 $xe^x = \cos x$	and $\ell = 0.01$

3. Suppose that f is a real-value function of a real variable. If f is continuous at a with $f(a) < M$ for some $M \in \mathbb{R}$, prove that there is an open interval I containing a such that

$$f(x) < M \text{ for all } x \in I.$$

4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty,$$

prove that f has a minimum on $\mathbb{R}$; i.e., there is an $x_m \in \mathbb{R}$ such that

$$f(x_m) = \inf_{x\in\mathbb{R}} f(x) < \infty.$$

5.3 Uniform continuity

Definition 5.3.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be **uniformly continuous on E** if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.3.2 *Prove that $f(x) = x$ is uniformly continuous on $(0, 1)$.*

Solution. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Let $x, a \in (0, 1)$ such that $|x - a| < \delta$. We obtain

$$|f(x) - f(a)| = |x - a| < \delta = \varepsilon.$$

Thus, f is uniformly continuous on $(0, 1)$.

Example 5.3.3 *Prove that $f(x) = x^2$ is uniformly continuous on $(0, 1)$.*

Solution. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2}$. Let $x, a \in (0, 1)$ such that $|x - a| < \delta$.
Then $|x + a| \leq |x| + |a| < 1 + 1 = 2$. We obtain

$$|f(x) - f(a)| = |x^2 - a^2| = |x + a||x - a| < 2\delta = \varepsilon.$$

Thus, f is uniformly continuous on $(0, 1)$.

Theorem 5.3.4 (**Uniform continuity of linear function**) *A Linear function is uniformly continuous on $\mathbb{R}$.*

Proof. Let m, c be contants and $f(x) = mx + c$ where $x \in \mathbb{R}$.
Let $\varepsilon > 0$. Then $|m| + 1 > 0$. Choose $\delta = \dfrac{\varepsilon}{|m| + 1} > 0$. Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.
We obtain by $\dfrac{|m|}{|m| + 1} < 1$ that

$$\begin{aligned} |f(x) - f(a)| &= |(mx + c) - (ma + c)| = |m(x - a)| = |m||x - a| \\ &< |m|\delta = |m| \cdot \frac{\varepsilon}{|m| + 1} < 1 \cdot \varepsilon = \varepsilon. \end{aligned}$$

Thus, f is uniformly continuous on $\mathbb{R}$. □

Example 5.3.5 *Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.*

Solution. Suppose that f is uniformly continuous on $\mathbb{R}$.

Given $\varepsilon = 1$. There is a $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x, a \in \mathbb{R} \quad \text{imply} \quad |f(x) - f(a)| < 1. \tag{5.5}$$

Choose $x = \dfrac{1}{\delta}$ and $a = \dfrac{1}{\delta} + \dfrac{\delta}{2}$. Then $|x - a| = \left|\dfrac{1}{\delta} - \left(\dfrac{1}{\delta} + \dfrac{\delta}{2}\right)\right| = \dfrac{\delta}{2} < \delta$ which satisfies (5.5).
We have $|f(x) - f(a)| < 1$ but

$$|f(x) - f(a)| = \left|x^2 - a^2\right| = |(x-a)(x+a)| = \left|\frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right)\right| = 1 + \frac{\delta^2}{4} > 1.$$

It is contradiction. Hence, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Theorem 5.3.6 *Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.*

Proof. Suppose to the contrary that f is continuous but not uniformly continuos on I.

Then there is an $\varepsilon_0 > 0$ such that

$$\text{for all } \delta > 0,\ |x - a| < \delta \text{ and } x, a \in I \text{ and } |f(x) - f(a)| \geq \varepsilon_0.$$

Set $\delta = \dfrac{1}{n}$. Then $x_n, y_n \in I$ such that $|x_n - y_n| < \dfrac{1}{n}$ and

$$|f(x_n) - f(y_n)| \geq \varepsilon_0, \quad \text{for } n \in \mathbb{N}. \tag{5.6}$$

Then sequence $\{x_n\}$ and $\{y_n\}$ are bounded. By The Bolzano-Weierstrass Theorem, $\{x_n\}$ has a subsequence, say x_{n_k}, that converges, as $k \to \infty$, to some $x \in I$. Similarly, $\{y_n\}$ has a subsequence, say y_{n_j}, that converges, as $j \to \infty$, to some $y \in I$. Since $x_{n_j} \to x$ as $j \to \infty$ and f is continuous, it follows by the Comparison Theorem from (5.6) that

$$\lim_{j\to\infty} |f(x_{n_j}) - f(y_{n_j})| \geq \varepsilon_0$$

$$|f(x) - f(y)| \geq \varepsilon_0 > 0$$

So, $f(x) \neq f(y)$. But $|x_n - y_n| < \dfrac{1}{n}$ for all $n \in \mathbb{R}$, so Theorem 1.3.10 implies that $x = y$. Thus, $f(x) = f(y)$, a contradiction. $\square$

Theorem 5.3.7 *Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, then $f(x_n)$ is Cauchy.*

Proof. Assume that $f : E \to \mathbb{R}$ is uniformly continuous and x_n is a Cauchy in E.
Let $\varepsilon > 0$. There is a $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon. \tag{5.7}$$

There is an $\mathbb{N}$ such that

$$n, m \geq N \quad \text{implies} \quad |x_n - x_m| < \delta.$$

For each $n, m \geq N$ such that $|x_n - x_m| < \delta$ it satisfies (5.7) that we have

$$|f(x_n) - f(x_m)| < \varepsilon.$$

Therefore, $f(x_n)$ is Cauchy. □

Exercises 5.3

1. Use Definition to prove that each of the following functions is uniformly continuous on $(0, 1)$.

   1.1 $f(x) = x^3$

   1.2 $f(x) = x^2 - x$

   1.3 $f(x) = \dfrac{1}{x+1}$

2. Prove that each of the following functions is uniformly continuous on $(0, 1)$.

   2.1 $f(x) = (x+1)^2$

   2.2 $f(x) = \dfrac{x^3-1}{x-1}$

   2.3 $f(x) = x\sin(\frac{1}{x})$

   2.4 $f(x)$ is any polynomial

   2.5 $f(x) = \dfrac{\sin x}{x}$

   2.6 $f(x) = x^2 \ln x$

3. Prove that $f(x) = \dfrac{1}{x^2+1}$ is uniformly continuous on $\mathbb{R}$.

4. Find all real α such that $x^\alpha \sin(\frac{1}{x})$ is uniformly continuous on the open interval $(0, 1)$.

5. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and there is an $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to \infty$. Prove that f is uniformly continuous on $[0, \infty)$.

6. Let I be a bounded interval. Prove that if $f : I \to \mathbb{R}$ is is uniformly continuous on I, then f is bounded on I.

7. Prove that (6) may be false if I is unbounded or if f is merely continuous.

8. Suppose that $\alpha \in \mathbb{R}$, E is nonempty subset of $\mathbb{R}$, and $f, g : E \to \mathbb{R}$ are uniformly continuous on E.

   8.1 Prove that $f + g$ and αf are uniformly continuous on E.

   8.2 Suppose that f, g are bounded on E. Prove that fg is uniformly continuous on E.

   8.3 Show that there exist functions f, g uniformly continuous on $\mathbb{R}$ such that fg is not uniformly continuous on $\mathbb{R}$.

9. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Chapter 6

Differentiability on $\mathbb{R}$

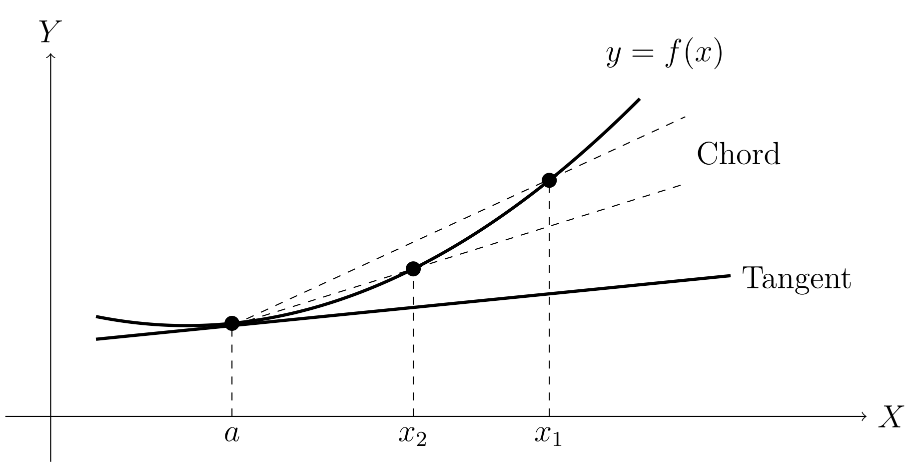
6.1 The Derivative

Definition 6.1.1 *A real function f is siad to be **differentiable** at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

*exists. In this case $f'(a)$ is called the **derivative** of f at a.*

You may recall that the graph of $y = f(x)$ has a **tangent line** at the point $(a, f(a))$ if and only if f has a derivative at a, in which case the slope of that tangent line is $f'(a)$. Suppose that f is differentiable at a. A **secant line** of the graph $y = f(x)$ is a line passing through at least two points on the graph, an a **chord** is a line segment that runs from one point on the graph to another.



Let $x = a + h$ and observe that the slope of the chord (chord function : $F(x)$) passing through the points $(x, f(x))$ and $(a, f(a))$ is given by

$$F(x) := \frac{f(x) - f(a)}{x - a}, \quad x \neq a.$$

Now, since $x = a + h$, $f'(a)$ becomes

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 6.1.2 *Let $f(x) = x^2$ where $x \in \mathbb{R}$. Find $f'(1)$*

Solution. We consider

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1}(x + 1) = 2.$$

Thus, f is differentiable at 1 and $f'(1) = 2$.

Example 6.1.3 *Show that the function*

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at the origin.

Solution. Consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos(\frac{1}{x})}{x} = \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$$

By Example 4.1.18, $f'(0) = 0$. Thus, f is differentiable at the origin.

Example 6.1.4 *Show that the function*

$$f(x) = \begin{cases} x \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not differentiable at the origin.

Solution. We consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \cos(\frac{1}{x})}{x} = \lim_{x \to 0} \cos\left(\frac{1}{x}\right).$$

By Example 4.1.12, the limit does not exist. Thus, f is not differentiable at the origin.

Theorem 6.1.5 *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at a if and only if there is a function T of the form $T(x) := mx$ such that*

$$\lim_{h\to 0}\left|\frac{f(a+h)-f(a)-T(h)}{h}\right| = 0.$$

Proof. Assume that f is differentiable at a. Then $f'(a)$ exists. Choose $m := f'(a)$. We obtain

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)-T(h)}{h} &= \lim_{h\to 0}\frac{f(a+h)-f(a)-mh}{h}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - m\\ &= f'(a)-f'(a) = 0\end{aligned}$$

Conversely, assume that $\lim_{h\to 0}\left|\frac{f(a+h)-f(a)-T(h)}{h}\right| = 0$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - m &= \lim_{h\to 0}\frac{f(a+h)-f(a)-mh}{h}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)-T(h)}{h} = 0.\end{aligned}$$

So, $f'(a) = m$. Thus, f is differentiable at a. □

Theorem 6.1.6 *If f is differentiable at a, then f is continuous at a.*

Proof. Assume that f is differentiable at a. Then $f'(a)$ exists. For $x \neq a$, we have

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}\cdot(x-a).$$

Taking limit $x \to a$, we obtain

$$\lim_{x\to a} f(x)-f(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) = f'(a)\cdot 0 = 0$$

So, $f(x) \to f(a)$ as $x \to a$. Hence, f is continuous at a. □

Example 6.1.7 *Show that $f(x) = |x|$ is continuous at 0 but not differentiable there.*

Solution. We see that

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$$

does not exist by Example 4.2.3. Thus, f is not differentiable at 0 but it easy to prove that f continuous at 0.

DIFFERENTIABLE ON INTERVAL.

Definition 6.1.8 *Let I be an interval and $f : I \to \mathbb{R}$ be a function. f is said to be* ***differentiable on I*** *if and only if f is differentiable at a for every $a \in I$*

Example 6.1.9 *Show that the function $f(x) = x^2$ is differentiable on $\mathbb{R}$.*

Solution. Let $a \in \mathbb{R}$. Then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a) = 2a.$$

Thus, f is is differentiable at a and $f'(a) = 2a$, i.e., $f'(x) = 2x$ for all $x \in \mathbb{R}$.

Theorem 6.1.10 *Let $n \in \mathbb{N}$. If $f(x) = x^n$, then f is differentiable on $\mathbb{R}$ and*

$$f'(x) = nx^{n-1}.$$

Proof. Use Binomial formula, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0}\frac{\left[x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n\right] - x^n}{h} \\ &= \lim_{h\to 0}\frac{h\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right]}{h} \\ &= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right] = \binom{n}{1}x^{n-1} = nx^{n-1}. \end{aligned}$$

□

Theorem 6.1.11 *Every constant function is differentiable on $\mathbb{R}$ and its value equals to zero.*

Proof. Let $f(x) = c$ where c is a constant. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

Thus, f is differentiable on $\mathbb{R}$ and $f'(x) = 0$. □

Example 6.1.12 *Show that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and $f'(x)$.*

Solution. Let $a > 0$

$$\begin{aligned}\lim_{x\to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \\ &= \lim_{x\to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\ &= \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}\end{aligned}$$

Thus f is is differentiable on $(0, \infty)$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$ for all $x > 0$.

Example 6.1.13 *Show that $f(x) = |x|$ is differentiable on $[0, 1]$ and $[-1, 0]$ but not on $[-1, 1]$.*

Solution. Consider $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Then f is differentiable on $(-\infty, 0) \cup (0, \infty)$ and

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Since f is not differentiable at 0, f is not differentiable on $[-1, 1]$. We see that

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1.$$

We conclude that f is not differentiable on $[-1, 0]$ and $[0, 1]$.

Exercises 6.1

1. For each of the following real functions, use definition directly to prove that $f'(a)$ exists.

   1.1 $f(x) = x^3, \quad a \in \mathbb{R}$

   1.2 $f(x) = \dfrac{1}{x}, \quad a \neq 0$

   1.3 $f(x) = x^2 + x + 2, \quad a \in \mathbb{R}$

   1.4 $f(x) = \dfrac{1}{\sqrt{x}}, \quad a > 0$

2. Prove that $f(x) = x|x|$ is differentiable on $\mathbb{R}$.

3. Let I be an open interval that contains 0 and $f : I \to \mathbb{R}$. If there exists an $\alpha > 1$ such that

$$|f(x)| \leq |x|^\alpha \text{ for all } x \in I,$$

   prove that f is differentiable at 0. What happens when $\alpha = 1$?

4. Suppose that $f : (0, \infty) \to \mathbb{R}$ satisfies $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$ for all $x, y \in (0, \infty)$ and $f(1) = 0$.

   4.1 Prove that f is continuous on $(0, \infty)$ if and only if f is continuous at 1.

   4.2 Prove that f is differentiable on $(0, \infty)$ if and only if f is differentiable at 1.

   4.3 Prove that if f is differentiable at 1, then $f'(x) = \dfrac{f'(1)}{x}$ for all $x \in (0, \infty)$.

5. Suppose that $f_\alpha(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that $f_\alpha(x)$ is continuous at $x = 0$ when $\alpha > 0$ and differentiable at $x = 0$ when $\alpha > 1$. Graph these functions for $\alpha = 1$ and $\alpha = 2$ and give a geometric interpretation of your results.

6. Prove that if $f(x) = x^\alpha$ where $\alpha = \frac{1}{n}$ for somw $n \in \mathbb{N}$, then $y = f(x)$ is differentiable on $f'(x) = \alpha x^{\alpha - 1}$ for every $x \in (0, \infty)$.

7. Given $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$. Show that

   7.1 $(\sin x)' = \cos x$

   7.2 $(\cos x)' = -\sin x$

8. f is a constant function on I if and only if $f'(x) = 0$ for every $x \in I$.

6.2 Differentiability theorem

Theorem 6.2.1 (**Additive Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $f+g$ is differentiable at a. In fact,*

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof. Assume that f and g are differentiable at a. Then

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h} \\ &= \lim_{h\to 0} \frac{[f(a+h)-f(a)]+[g(a+h)-g(a)]}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\ &= f'(a)+g'(a)\end{aligned}$$

Thus, $(f+g)'(a) = f'(a) + g'(a)$. □

Theorem 6.2.2 (**Scalar Multiplicative Rule**) *Let f be a real function and $\alpha \in \mathbb{R}$. If f is differentiable at a, then αf is differentiable at a. In fact,*

$$(\alpha f)'(a) = \alpha f'(a).$$

Proof. Assume that f is differentiable at a. Then

$$\begin{aligned}(\alpha f(a))' &= \lim_{h\to 0} \frac{\alpha f(a+h)-\alpha f(a)}{h} \\ &= \alpha \cdot \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \alpha f'(a).\end{aligned}$$

Thus, $(\alpha f)'(a) = \alpha f'(a)$. □

Theorem 6.2.3 (**Product Rule**) *Let f and g be real functions. If f and g are differentiable at a, then fg is differentiable at a. In fact,*

$$(fg)'(a) = g(a)f'(a) + f(a)g'(a).$$

Proof. Assume that f and g are differentiable at a. Then

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0} \frac{(fg)(a+h) - (fg)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a) + f(a+h)g(a) - f(a+h)g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)[g(a+h) - g(a)] + g(a)[f(a+h) - f(a)]}{h} \\
&= \lim_{h\to 0} f(a+h) \cdot \frac{g(a+h) - g(a)}{h} + \lim_{h\to 0} g(a) \cdot \frac{f(a+h) - f(a)}{h} \\
&= f(a)g'(a) + g(a)f'(a).
\end{aligned}$$

Thus, $(fg)'(a) = g(a)f'(a) + f(a)g'(a)$. □

Theorem 6.2.4 (**Quotient Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $\frac{f}{g}$ is differentiable at a when $g(a) \neq 0$. In fact,*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof. Assume that f and g are differentiable at a when $g(a) \neq 0$. Then

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{h\to 0} \frac{\frac{f}{g}(a+h) - \frac{f}{g}(a)}{h} = \lim_{h\to 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a+h)} + \frac{f(a)}{g(a+h)} - \frac{f(a)}{g(a)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{g(a+h)}[f(a+h) - f(a)] + f(a)\left[\frac{1}{g(a+h)} - \frac{1}{g(a)}\right]}{h} \\
&= \lim_{h\to 0} \frac{\frac{g(a)}{g(a)g(a+h)}[f(a+h) - f(a)] - f(a)\left[\frac{g(a+h)-g(a)}{g(a+h)g(a)}\right]}{h} \\
&= \lim_{h\to 0} \frac{g(a)\left[\frac{f(a+h)-f(a)}{h}\right] - f(a)\left[\frac{g(a+h)-g(a)}{h}\right]}{g(a)g(a+h)} = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}.
\end{aligned}$$

□

Example 6.2.5 *Let f and g be differentiable at 1 with $f(1) = 1, g(1) = 2$ and $f'(1) = 3, g'(1) = 4$. Evaluate the following derivatives.*

1. $(f + g)'(1) = f'(1) + g'(1) = 3 + 4 = 7.$
2. $(2f)'(1) = 2f'(1) = 2 \cdot 3 = 6.$
3. $(fg)'(1) = f(1)g'(1) + f'(1)g(1) = 1 \cdot 4 + 3 \cdot 2 = 10.$
4. $\left(\dfrac{f}{g}\right)'(1) = \dfrac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \dfrac{2 \cdot 3 - 1 \cdot 4}{2^2} = \dfrac{1}{2}.$

Theorem 6.2.6 (**Chain Rule**) *Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. Assume that f is differentiable at a and g is differentiable at $f(a)$.

Then $f'(a)$ and $g'(f(a))$ exist. We consider

$$\begin{aligned} f(x) &= \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a), \qquad x \neq a \\ g(y) &= \frac{g(y) - g(f(a))}{y - f(a)} \cdot (y - f(a)) + g(f(a)), \qquad y \neq f(a) \end{aligned} \tag{6.1}$$

Since f is continuous at a, substitue $y = f(x)$ in (6.1) to write

$$\begin{aligned} g(f(x)) &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot (f(x) - f(a)) + g(f(a)) \\ g(f(x)) &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \cdot (x - a) + g(f(a)) \\ \frac{g(f(x)) - g(f(a))}{x - a} &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ (g \circ f)'(a) &= g'(f(a)) \cdot f'(a) \end{aligned}$$

□

Example 6.2.7 *Let f and g be differentiable on $\mathbb{R}$ with $f(0) = 1, g(0) = -1$ and $f'(0) = 2$, $g'(0) = -2$,$f'(-1) = 3$, $g'(1) = 4$. Evaluate each of the following derivatives.*

1. $(f \circ g)'(0) = f'(g(0))g'(0) = f'(-1) \cdot g'(0) = 3(-2) = -6.$

2. $(g \circ f)'(0) = g'(f(0))f'(0) = g'(1) \cdot f'(0) = 4(2) = 8.$

Example 6.2.8 *Let $f(x) = \sqrt{x^2+1}$. Use the Chain Rule to show that $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$.*

Solution. Let $g(x) = \sqrt{x}$ and $h(x) = x^2 + 1$. We have

$$g'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad h'(x) = 2x.$$

By Chain Rule,

$$\begin{aligned} f'(x) = (g \circ h)'(x) &= g'(h(x))h'(x) \\ &= \frac{1}{2\sqrt{h(x)}} \cdot h'(x) \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

Exercises 6.2

1. For each of the following functions, find all x for which $f'(x)$ exists and find a formula for f'.

   1.1 $f(x) = \dfrac{x^3 - 2x^2 + 3x}{\sqrt{x}}$

   1.2 $f(x) = \dfrac{1}{x^2 + x - 1}$

   1.3 $f(x) = x|x|$

   1.4 $f(x) = |x^3 + 2x^2 - x - 2|$

2. Let f and g be differentiable at 2 and 3 with $f'(2) = a$, $f'(3) = b$, $g'(2) = c$ and $g'(3) = d$, If $f(2) = 1$, $f(3) = 2$,$g(2) = 3$ and $g(3) = 4$. Evaluate each of the following derivatives.

   2.1 $(fg)'(2)$

   2.2 $\left(\frac{f}{g}\right)'(3)$

   2.3 $(g \circ f)'(3)$

   2.4 $(f \circ g)'(2)$

3. If f, g and h is differentiable at a, prove that fgh is differentiable at a and
$$(fgh)'(a) = f'(a)g(a)h(a) + f(a)g'(a)h(a) + f(a)g(a)h'(a).$$

4. Let $f(x) = (x-1)(x-2)(x-3)\cdots(x-2565)$. Find $f'(2565)$

5. Prove that if $f(x) = x^{\frac{m}{n}}$ for some $n, m \in \mathbb{N}$, then $y = f(x)$ is differentiable and satisfies $ny^{n-1}y' = mx^{m-1}$ for every $x \in (0, \infty)$.

6. (**Power Rule**) Prove that $f(x) = x^q$ for some $q \in \mathbb{Q}$, then f is differentiable and $f'(x) = qx^{q-1}$ for every $x \in (0, \infty)$.

7. **(Reciprocal Rule)** Suppose that f is differentiable at a and $f(a) \neq 0$.

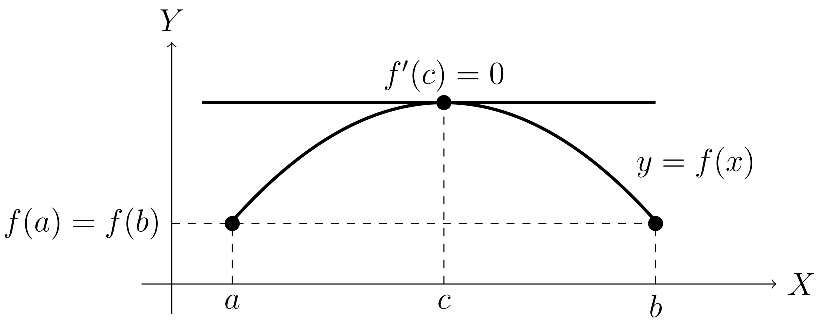
   7.1 Show that for h sufficiently small, $f(a+h) \neq 0$.

   7.2 Use Definition 6.1.1 directly, prove that $\dfrac{1}{f(x)}$ is differentiable at $x = a$ and
$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{f^2(a)}.$$

8. Suppose hat $n \in \mathbb{N}$ and f, g are real functions of a real variable whose nth derivatives $f^{(n)}, g^{(n)}$ exist at a point a. Prove Leibniz's generalization of the Product Rule:
$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a).$$

6.3 Mean Value Theorem

Lemma 6.3.1 (**Rolle's Theorem**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.*



Proof. Let $a \neq b$ such that f is continuous on $[a, b]$ and differentiable on (a, b).

Assume that $f(a) = f(b)$. By EVT, f has a finite maximum M and a finite minimum m on $[a, b]$.

Case $M = m$. Then f is a constant function. Thus, $f'(x) = 0$ for all $x \in (a, b)$.

Case $M \neq m$. Since $f(a) = f(b)$, there is a $c \in (a, b)$ such that $f(c) = M$. We have

$$f(c + h) \leq f(c) \quad \text{for all } h \text{ that satisfy } c + h \in (a, b).$$

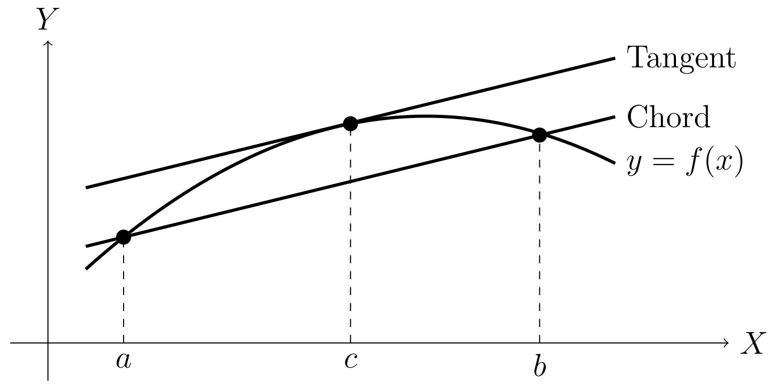
In the case $h > 0$ this implies that

$$f'(c) = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \leq 0,$$

and in this case $h < 0$ this implies that

$$f'(c) = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \geq 0.$$

It follows that $f'(c) = 0$. $\square$



Theorem 6.3.2 (**Mean Value Theorem (MVT)**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is an $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let $a \neq b$ such that f is continuous on $[a, b]$ and differentiable on (a, b). We set

$$h(x) = f(x)(b - a) - x[f(b) - f(a)] \quad \text{for } x \in [a, b].$$

Then h is continuous on $[a, b]$ and differentiable (a, b),

$$h'(x) = f'(x)(b - a) - [f(b) - f(a)].$$

We obtain

$$\begin{aligned} h(a) &= f(a)(b - a) - a[f(b) - f(a)] = bf(a) - af(b) \\ &= bf(a) - af(b) + bf(b) - bf(b) = f(b)(b - a) - b[f(b) - f(a)] = h(b). \end{aligned}$$

By the Rolle's Theorem, there is a $c \in (a, b)$ such that $h'(c) = 0$, i.e.,

$$f'(c)(b - a) - [f(b) - f(a)] = 0.$$

Hence, $f(b) - f(a) = f'(c)(b - a)$. □

Example 6.3.3 *Prove that*

$$\sin x \leq x \quad \textit{for all } x > 0.$$

Solution. Let $a > 0$ and define $f(x) = \sin x$ where $x \in [0, a]$. Then f is continuous on $[0, a]$ and $f(x)$ is differentiable and $f'(x) = \cos x$ for every $x \in (0, a)$.
By the MVT, there is a $c \in (0, a)$ such that

$$f(a) - f(0) = f'(c)(a - 0)$$
$$\sin a - 0 \cos c \cdot a$$
$$\sin a = a \cos c$$

From $\cos c \leq 1$ and $a > 0$, $a \cos c \leq a$, it implies that $\sin a < a$. Therefore,

$$\sin x \leq x \quad \text{for all } x > 0.$$

Example 6.3.4 *Prove that*

$$1 + x \leq e^x \quad \textit{for all } x > 0.$$

Solution. Let $a > 0$ and define $f(x) = e^x - x - 1$ where $x \in [0, a]$. Then f is continuous on $[0, a]$ and $f(x)$ is differentiable and $f'(x) = e^x - 1$ for every $x \in (0, a)$.
By the MVT, there is a $c \in (0, a)$ such that

$$f(a) - f(0) = f'(c)(a - 0)$$
$$(e^a - a - 1) - 0 = (e^c - 1)a$$
$$e^a - a - 1 = (e^c - 1)a$$

Since $c \geq 0$, $e^c \geq 1$ or $e^c - 1 \geq 0$. From $a > 0$, it implies that $(e^c - 1)a \geq 0$ which leads to $e^a - a - 1 \geq 0$ Therefore,

$$1 + x \leq e^x \quad \text{for all } x > 0.$$

Example 6.3.5 (**Bernoulli's Inequality**) *Let* $0 < \alpha \le 1$ *and* $\delta \ge -1$. *Prove that*

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

Proof. Let $0 < \alpha \le 1$ and $\delta \ge -1$. Define $f(x) = x^\alpha$ where $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ and $f(x)$ is differentiable and

$$f'(x) = \alpha x^{\alpha-1} \quad \text{for every } x \in \mathbb{R}.$$

Case $-1 \le \delta \le 0$. By the MVT, there is a $c \in (1+\delta, 1)$ such that

$$\begin{aligned} f(1) - f(1+\delta) &= f'(c)[1-(1+\delta)] \\ 1-(1+\delta)^\alpha &= -\delta\alpha c^{\alpha-1} \\ (1+\delta)^\alpha - 1 &= \delta\alpha c^{\alpha-1} \end{aligned}$$

Since $0 < \alpha \le 1$, $-1 < \alpha - 1 \le 0$. From $0 \le 1+\delta < c < 1$, it implies that $c^{\alpha-1} \ge c^0 = 1$. Since $\delta \le 0$ and $\alpha > 0$, $\delta\alpha \le 0$ which leads to $\delta\alpha c^{\alpha-1} \le \alpha\delta$. Thus,

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

Case $\delta > 0$. By the MVT, there is a $c \in (1, 1+\delta)$ such that

$$\begin{aligned} f(1+\delta) - f(1) &= f'(c)[(1+\delta)-1] \\ (1+\delta)^\alpha - 1 &= \delta\alpha c^{\alpha-1} \end{aligned}$$

Since $0 < \alpha \le 1$, $-1 < \alpha - 1 \le 0$. From $c > 1$, it implies that $c^{\alpha-1} \le c^0 = 1$. Since $\delta > 0$ and $\alpha > 0$, $\delta\alpha > 0$ which leads to $\delta\alpha c^{\alpha-1} \le \alpha\delta$. Thus,

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

We conclude that $(1+\delta)^\alpha \le 1 + \alpha\delta$ for $0 < \alpha \le 1$ and $\delta \ge -1$. $\square$

Theorem 6.3.6 (**Generalized Mean Value Theorem**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is an $c \in (a, b)$ such that*

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)].$$

Proof. Let $a \neq b$ such that f and g are continuous on $[a, b]$ and differentiable on (a, b). We set

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)] \quad \text{for } x \in [a, b].$$

Then h is continuous on $[a, b]$ and differentiable (a, b),

$$h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)].$$

We obtain

$$\begin{aligned}
h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\
&= f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) \\
&= f(a)g(b) - g(a)f(b) \\
&= f(a)g(b) - g(a)f(b) + g(b)f(b) - g(b)f(b) \\
&= [f(b)g(b) - f(b)g(a)] + [g(b)f(a) - g(b)f(b)] \\
&= f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] \\
&= h(b).
\end{aligned}$$

By the Rolle's Theorem, there is a $c \in (a, b)$ such that $h'(c) = 0$, i.e.,

$$f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0.$$

Hence, $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$. □

Theorem 6.3.7 (**L'Hôspital's Rule**) *Let a be an extended real number and I be an open interval that either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I\backslash\{a\}$, and $g(x) \neq 0 \neq g'(x)$ for all $x \in I\backslash\{a\}$. Suppose further that*

$$A := \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

is either 0 or ∞. If

$$B := \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. We will use the SCL to prove that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = B.$$

Let $x_k \in I\backslash\{a\}$ such that $x_k \to a$ as $k \to \infty$. Note that if g' is never zero on $I\backslash\{a\}$. By the MVT, for $x, y < a$ or $x, y > a$ there is a $c \in (x, y)$ such that

$$g(x) - g(y) = g'(c)(y - x) \neq 0 \quad \text{for all } x \neq y.$$

We suppose for simplicity that $B \in \mathbb{R}$. (For case $B = \pm\infty$, see Exercise.)

Case 1. $A = 0$ and $a \in \mathbb{R}$. Extend f and g to $I \cup \{a\}$ by $f(a) = 0 = g(a)$. By hypothesis, f and g are continuous on $I \cup \{a\}$ and differentiable on $I\backslash\{a\}$. By the Generalized Mean Value Theorem, there is a c_k between x_k and a such that

$$\begin{aligned} g'(c_k)[f(x_k) - f(a)] &= f'(c_k)[g(x_k) - g(a)] \\ g'(c_k)[f(x_k) - 0] &= f'(c_k)[g(x_k) - 0] \\ \frac{f(x_k)}{g(x_k)} &= \frac{f'(c_k)}{g'(c_k)} \end{aligned}$$

From $x_k < c_k < a$ or $a < c_k < x_k$, it implies $c_k \to a$ as $k \to \infty$ by the Squeeze Theorem.

We conclude that

$$\lim_{k\to\infty} \frac{f(x_k)}{g(x_k)} = \lim_{k\to\infty} \frac{f'(c_k)}{g'(c_k)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = B.$$

Case 2. $A = \pm\infty$ and $a \in \mathbb{R}$. We suppose by symmetry that $A = \infty$. For each $k, n \in \mathbb{N}$, apply the Generalized Mean Value Theorem, there is a $c_{k,n}$ between x_k and x_n such that

$$f(x_n) - f(x_k) = \frac{f'(c_{k,n})}{g'(c_{k,n})} \cdot [g(x_n) - g(x_k)].$$

We obtain

$$\begin{aligned}\frac{f(x_n)}{g(x_n)} - \frac{f(x_k)}{g(x_n)} &= \frac{f(x_n) - f(x_k)}{g(x_n)} = \frac{1}{g(x_n)} \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})} \cdot [g(x_n) - g(x_k)] \\ &= \frac{f'(c_{k,n})}{g'(c_{k,n})} - \frac{g(x_k)}{g(x_n)} \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})}.\end{aligned}$$

It leads to

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_k)}{g(x_n)} + \frac{f'(c_{k,n})}{g'(c_{k,n})} - \frac{g(x_k)}{g(x_n)} \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})}. \tag{6.2}$$

Since $A = \infty$, it is clear that $\dfrac{1}{g(x_n)} \to 0$ as $n \to \infty$, and since $c_{n,k}$ lies between x_k and x_n, it also clear that $c_{k,n} \to a$ as $k, n \to \infty$ by the Squeeze Theorem. Thus, the limit of $\dfrac{f'(c_{k,n})}{g'(c_{k,n})}$ exists as $n \to \infty$ and fixed $k \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty} \frac{f(x_k)}{g(x_n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{g(x_k)}{g(x_n)} = 0.$$

Hence, (6.2) becomes to

$$\begin{aligned}\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} &= \lim_{n\to\infty} \left[\frac{f(x_k)}{g(x_n)} + \frac{f'(c_{k,n})}{g'(c_{k,n})} - \frac{g(x_k)}{g(x_n)} \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})}\right] \\ &= 0 + \lim_{n\to\infty} \frac{f'(c_{k,n})}{g'(c_{k,n})} - 0 \cdot \lim_{n\to\infty} \frac{f'(c_{k,n})}{g'(c_{k,n})} \\ &= \lim_{n\to\infty} \frac{f'(c_{k,n})}{g'(c_{k,n})} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = B.\end{aligned}$$

Case 3. $a = \pm\infty$. We suppose by symmetry that $a = \infty$. Choose $c > 0$ such that $(c, \infty) \subset I$. For each $y \in (0, \frac{1}{c})$, set

$$\phi(y) = f\left(\frac{1}{y}\right) \quad \text{and} \quad \varphi(y) = g\left(\frac{1}{y}\right).$$

By the Chain Rule,

$$\frac{\phi'(y)}{\varphi'(y)} = \frac{f'(\frac{1}{y}) \cdot (-\frac{1}{y^2})}{g'(\frac{1}{y}) \cdot (-\frac{1}{y^2})} = \frac{f'(\frac{1}{y})}{g'(\frac{1}{y})}.$$

Thus, for $x = \frac{1}{y} \in (c, \infty)$, we have $\dfrac{\phi'(y)}{\varphi'(y)} = \dfrac{f'(x)}{g'(x)}$. Since $x \to \infty$ if and only if $y = \frac{1}{x} \to 0^+$, it follows that ϕ and φ satisfy the hypothesis of Case 1 or 2 for $a = 0$ and $I = (0, \frac{1}{c})$. In particular,

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{y\to 0^+} \frac{\phi'(y)}{\varphi'(y)} = \lim_{y\to 0^+} \frac{\phi(y)}{\varphi(y)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

$\square$

Given $(\ln x)' = \dfrac{1}{x}$ for $x > 0$ and $(e^x)' = e^x$ for all $x \in \mathbb{R}$.

Example 6.3.8 *Use L'H****ô****spital's Rule to prove that* $\lim\limits_{x\to 0} \dfrac{x}{e^x - 1} = 1.$

Solution. We see that

$$\lim_{x\to 0} x = 0 = \lim_{x\to 0} e^x - 1.$$

By L'H***ô***spital's Rule, it follows that

$$\lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{(x)'}{(e^x - 1)'} = \lim_{x\to 0} \frac{1}{e^x} = 1.$$

Example 6.3.9 *Use L'H****ô****spital's Rule to find* $\lim\limits_{x\to 0^+} x \ln x.$

Solution. We see that

$$\lim_{x\to 0^+} \ln x = \infty = \lim_{x\to 0^+} \frac{1}{x}.$$

By L'H***ô***spital's Rule, it follows that

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(x^{-1})'} \\ &= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0.\end{aligned}$$

Example 6.3.10 *Use L'H****ô****spital's Rule to find* $L = \lim\limits_{x\to 1^-} (\ln x)^{1-x}.$

Solution. We see that

$$\lim_{x\to 1^-} \ln(\ln x) = -\infty = \lim_{x\to 1^-} \frac{1}{1-x}.$$

Since $\ln x$ is continuous on $(0, \infty)$, by L'H***ô***spital's Rule we have

$$\begin{aligned}\ln L = \ln \lim_{x\to 1^-} (\ln x)^{1-x} &= \lim_{x\to 1^-} \ln(\ln x)^{1-x} = \lim_{x\to 1^-} (1-x)\ln(\ln x) = \lim_{x\to 1^-} \frac{(\ln(\ln x))'}{(\frac{1}{1-x})'} \\ &= \lim_{x\to 1^-} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{(1-x)^2}} = \lim_{x\to 1^-} \frac{(1-x)^2}{x\ln x}\end{aligned}$$

Apply again L'H***ô***spital's Rule, we obtain

$$\begin{aligned}\ln L &= \lim_{x\to 1^-} \frac{[(1-x)^2]'}{[x\ln x]'} = \lim_{x\to 1^-} \frac{-2(1-x)}{\ln x + 1} = 0 \\ L &= e^0 = 1.\end{aligned}$$

Hence, $L = \lim\limits_{x\to 1^-} (\ln x)^{1-x} = 1.$

Exercises 6.3

1. Use the Mean Value Theorem to prove that each of the following inequalities.

   1.1 $\sqrt{2x+1} < 1+x$ for all $x > 0$

   1.2 $\ln x \le x-1$ for all $x > 1$

   1.3 $7(x-1) < e^x$ for all $x > 2$

   1.4 $\cos x - 1 \le x$ for all $x > 0$

   1.5 $\ln x + 1 \le \dfrac{x^2+1}{2}$ for all $x > 1$

   1.6 $\dfrac{x-1}{x} \le \ln x$ for all $x > 1$

   1.7 $\sqrt{x} \ge x$ for all $x \in [0,1]$

   1.8 $\sqrt{x} \le x$ for all $x > 1$

   1.9 $\sin^2 x \le 2|x|$ for all $x \in \mathbb{R}$

   1.10 $\ln x \le \sqrt{x}$ for all $x > 1$

2. (**Bernoulli's Inequality**) Let $\alpha \ge 1$ and $\delta \ge -1$. Prove that

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

3. Use L'H**ô**spital's Rule to evaluate the following limits.

   3.1 $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{x}$

   3.2 $\lim\limits_{x\to 0^+} \dfrac{\cos x - e^x}{\ln(1+x^2)}$

   3.3 $\lim\limits_{x\to 0} \left(\dfrac{x}{\sin x}\right)^{\frac{1}{x^2}}$

   3.4 $\lim\limits_{x\to 0^+} x^x$

   3.5 $\lim\limits_{x\to 1} \dfrac{\ln x}{\sin(\pi x)}$

   3.6 $\lim\limits_{x\to \infty} x\left(\arctan x - \dfrac{\pi}{2}\right)$

   3.7 $\lim\limits_{x\to 0^-} \left(1+e^{-x}\right)^x$

   3.8 $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}}$

   3.9 $\lim\limits_{x\to \infty} x(e^{\frac{1}{x}} - 1)$

4. Show that the derivative of

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   exists and continuous on $\mathbb{R}$ with $f'(0) = 0$.

5. Suppose that f is differentiable on $\mathbb{R}$.

   5.1 If $f'(x) = 0$ for all $x \in \mathbb{R}$, prove that $f(x) = f(0)$ for all $x \in \mathbb{R}$

   5.2 If $f(0) = 1$ and $|f'(x)| \le 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \le |x| + 1$ for all $x \in \mathbb{R}$

   5.3 If $'(x) \ge 0$ for all $x \in \mathbb{R}$, prove that $a < b$ imply that $f(a) < f(b)$

6. Let f be differentiable on a nonempty, open interval (a,b) with f' bounded on (a,b). Prove that f is uniformly continuous on (a,b).

7. Let f be differentiable on (a,b), continuous on $[a,b]$, with $f(a) = f(b) = 0$. Prove that if $f'(c) > 0$ for some $c \in (a,b)$, then there exist $x_1, x_2 \in (a,b)$ such that $f'(x_1) > 0 > f'(x_2)$.

8. Let f be twice differentiable on (a,b) and let there be points $x_1 < x_2 < x_3$ in (a,b) such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there is a point $c \in (a,b)$ such that $f''(c) > 0$.

9. Let f be differentiable on $(0,\infty)$. If $L = \lim_{x\to\infty} f'(x)$ and $\lim_{n\to\infty} f(n)$ both exist and are finite, prove that $L = 0$.

10. Prove L'Hôspital's Rule for the case $B = \pm\infty$ by first proving that
$$\frac{g(x)}{f(x)} \to 0 \text{ when } \frac{f(x)}{g(x)} \to \pm\infty, \text{ as } x \to a.$$

11. Prove that the sequence $\left(1 + \dfrac{1}{n}\right)^n$ is increasing, as $n \to \infty$, and its limit e satisfies $2 < e \le 3$ and $\ln e = 1$.

6.4 Monotone function

Definition 6.4.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*

1. *f is said to be* ***increasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \leq f(x_2).$$

   f is said to be ***strictly increasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) < f(x_2).$$

2. *f is said to be* ***decreasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \geq f(x_2).$$

   f is said to be ***strictly decreasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) > f(x_2).$$

3. *f is said to be* ***monotone*** *on E if and only if f is either decreasing or increasing on E.*
   f is said to be ***strictly monotone*** *on E if and only if f is either strictly decreasing or strictly increasing on E.*

Example 6.4.2 *Show that $f(x) = x^2$ is strictly monotone on $[0, 1]$ and on $[-1, 0]$ but not monotone on $[-1, 1]$.*

Solution.

If $0 \leq x < y \leq 1$, then $x^2 < y^2$, i.e., $f(x) < f(y)$. Thus, f is strictly increasing on $[0, 1]$.
If $-1 \leq x < y \leq 0$, then $x^2 > y^2$, i.e., $f(x) > f(y)$. Thus, f is strictly decreasing on $[-1, 0]$.
We conclude that f is strictly monotone on $[0, 1]$ and on $[-1, 0]$.

Since f is increasing and decreasing on $[-1, 1]$, f is not monotone on $[-1, 1]$.

Theorem 6.4.3 *Let $f : I \to \mathbb{R}$ and $(a, b) \subseteq I$. Then*

1. *f is increasing on (a, b) if $f'(x) > 0$ for all $x \in (a, b)$*
2. *f is decreasing on (a, b) if $f'(x) < 0$ for all $x \in (a, b)$*
3. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Proof. Let $x, y \in (a, b)$ such that $x < y$. Then $y - x > 0$.
By the MVT, there is a $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x) > 0.$$

If $f'(x) > 0$ for all $x \in (a, b)$, $f'(c) > 0$. It follows that $f(y) > f(x)$. So, f is increasing on (a, b).
If $f'(x) < 0$ for all $x \in (a, b)$, $f'(c) < 0$. It follows that $f(y) < f(x)$. So, f is decreasing on (a, b).

Let $x \in [a, b]$. By the MVT, there is a $c \in (a, x)$ such that

$$f(x) - f(a) = f'(c)(x - a) = 0.$$

So, $f(x) = f(a)$ for all $x \in [a, b]$. We conclude that f is constant on $[a, b]$. □

Example 6.4.4 *Find each intervals of $f(x) = x^2 - 4x + 3$ that increasing and decreasing.*

Solution. We have $f'(x) = 2x - 4$. Consider

$$2x - 4 = f'(x) > 0 \quad \text{implies} \quad x > 2.$$

Thus, f is increasing on $(2, \infty)$.

$$2x - 4 = f'(x) < 0 \quad \text{implies} \quad x < 2.$$

Thus, f is increasing on $(-\infty, 2)$.

Theorem 6.4.5 *If f is 1-1 and continuous on an interval I, then f is strictly monotone on I and f^{-1} is continuous and strictly monotone on $f(I) := \{f(x) : x \in I\}$.*

Proof. Assume that f is 1-1 and continuous on an interval I. Let $a, b \in I$ such that

$$a < b \text{ implies either } f(a) < f(b) \text{ or } f(a) > f(b).$$

Suppose that f is not strictly monotone on I. Then there exist points $a, b, c \in I$ such that $a < c < b$ but $f(c)$ does not lie between $f(a)$ and $f(b)$. It follows that either $f(a)$ lie between $f(b)$ and $f(c)$ or $f(b)$ lie between $f(a)$ and $f(c)$. Hence by the IVT, there is an $x_1 \in (a, b)$ such that

$$f(x_1) = f(a) \quad \text{or} \quad f(x_1) = f(b).$$

Since f is 1-1, we conclude that either $x_1 = a$ or $x_2 = b$, a contradiction. Therefore, f is strictly monotone on I.

We may suppose that f is strictly increasing on I. Since f is 1-1 on I, apply Theorem 1.4.3 to verify that f^{-1} takes $f(I)$ onto I. We will show that f^{-1} is strictly increasing on $f(I)$. Suppose to the contrary that there exist $y_1, y_2 \in f(I)$ such that

$$y_1 < y_2 \quad \text{but} \quad f^{-1}(y_1) \geq f^{-1}(y_2).$$

Then $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ satisfy $x_1 \geq x_2$ and $x_1, x_2 \in I$. Since f is strictly increasing on I, it follows that $y_1 = f(x_1) \geq f(x_2) = y_2$, a contracdiction.
Thus, f^{-1} is strictly increasing on $f(I)$.

Since I is a interval, it easy to prove that $f(I)$ is also inverval. Fix $y_0 \in f(I)$ and $\varepsilon > 0$. Since f^{-1} is strictly increasing on $f(I)$, if y_0 is not right endpoint of $f(I)$, then $x_0 = f^{-1}(y_0)$ is not right endpoint of I. There is an $\varepsilon_0 > 0$ so small that $\varepsilon_0 < \varepsilon$ and $x_0 + \varepsilon_0 \in I$. Choose $\delta = f(x_0 + \varepsilon_0) - f(x_0)$ and suppose that $0 < y - y_0 < \delta$. The choice of δ implies that

$$y_0 < y < y_0 + \delta = f(x_0) + \delta = f(x_0 + \varepsilon_0).$$

Set $y = f^{-1}(x)$. Then $f(x_0) < f(x) < f(x_0 + \varepsilon_0)$. Since f is strictly increasing on I, it implies $x_0 < x < x_0 + \varepsilon_0$, i.e., $0 < x - x_0 < \varepsilon_0$. We conclude that

$$0 < f^{-1}(x) - f^{-1}(y_0) < \varepsilon.$$

So, $f^{-1}(y_0^+) = f^{-1}(y_0)$. A similar argument show that if y_0 is not a left endpoint of $f(I)$, $f^{-1}(y_0^-) = f^{-1}(y_0)$. Hence, f^{-1} is continuous on $f(I)$. □

Theorem 6.4.6 (**Inverse Function Theorem (IFT)**) *Let f be 1-1 and continuous on an open interval I. If $a \in f(I)$ and if $f'(f^{-1}(a))$ exists and is nonzero, then f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Proof. Let f be 1-1 and continuous on an open interval I. By Theorem 6.4.5, f is strictly monotone, say strictly increasing on I and f^{-1} exists, is continuous and strictly increasing on $f(I)$.
Assume that $a \in f(I)$ and $f'(f^{-1}(a))$ exists and is nonzero. Set $x_0 = f^{-1}(a) \in I$ and I is open, we can choose $c, d \in \mathbb{R}$ such that $x_0 \in (c, d) \subset I$. Then $a = f(x_0) \in (f(c), f(d)) \subset f(I)$.
We can choose $h \neq 0$ so small that $a + h \in f(I)$. i.e., $f^{-1}(a + h)$ exists. Set $x = f^{-1}(a + h)$ and observe that $f(x) - f(x_0) = a + h - a = h$. Since f^{-1} is continuous, $x \to x_0$ if and only if $h \to 0$. Therefore,

$$(f^{-1})'(a) = \lim_{h \to 0} \frac{f^{-1}(a+h) - f^{-1}(a)}{h} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(a))}.$$

□

Example 6.4.7 *Use the Inverse Function Theorem to find derivative of $f(x) = \arcsin x$*

Solution. Let $g(x) = \sin x$ where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then g is 1-1 and continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
We have $g'(x) = \cos x > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $g^{-1}(x) = \arcsin x = f(x)$.
By the IFT, we obtain

$$\begin{aligned} f'(x) = (\arcsin x)' = (g^{-1})'(x) &= \frac{1}{g'(g^{-1}(x))} \\ &= \frac{1}{g'(\arcsin x)} \\ &= \frac{1}{\cos(\arcsin x)} \\ &= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

Example 6.4.8 *Let $f(x) = x + e^x$ where $x \in \mathbb{R}$.*

1. *Show that f is 1-1 on $x \in \mathbb{R}$.*
2. *Use the result from 1 and the IFT to explain that f^{-1} differentiable on $\mathbb{R}$.*
3. *Compute $(f^{-1})'(2 + \ln 2)$.*

Solution.

1. *Proof.* Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x > y$. Then $x - y > 0$ and $e^x > e^y$. We obtain

$$\begin{aligned} e^y - e^x &< 0 < x - y \\ y + e^y &< x + e^x \\ f(y) &< f(x). \end{aligned}$$

   So, $f(x) \neq f(y)$. Therefore, f is injective in $\mathbb{R}$. □

2. Since f is 1-1, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By the IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

3. We see that $f'(x) = 1 + e^x$ and $f(\ln 2) = \ln 2 + 2$. So, $f^{-1}(2 + \ln 2) = \ln 2$.

   By the IFT, we obtain

$$(f^{-1})'(2 + \ln 2) = \frac{1}{f'(f^{-1}(2 + \ln 2))} = \frac{1}{f'(\ln 2)} = \frac{1}{1 + 2} = \frac{1}{3}.$$

Exercises 6.4

1. Find each intervals of the following functions that increasing and decreasing.

   1.1 $f(x) = 2x - x^2$

   1.2 $f(x) = x^3 - x^2 - x + 3$

   1.3 $f(x) = (x-1)^3(x-2)^4$

   1.4 $g(x) = xe^x$

   1.5 $g(x) = e^x - x$

   1.6 $g(x) = x^2e^{x^2}$

2. Find all $a \in \mathbb{R}$ such that $x^3 + ax^2 + 3x + 15$ is strictly increasing near $x = 1$.

3. Find all $a \in \mathbb{R}$ such that $ax^2 + 3x + 5$ is strictly increasing on the interval $(1, 2)$.

4. Find where $f(x) = 2|x-1| + 5\sqrt{x^2+9}$ is strictly increasing and where $f(x)$ is strictly decreasing.

5. Let f and g be 1-1 and continuous on $\mathbb{R}$. If $f(0) = 2$, $g(1) = 2$, $f'(0) = \pi$, and $g'(1) = e$, compute the following derivatives.

   5.1 $(f^{-1})'(2)$

   5.2 $(g^{-1})'(2)$

   5.3 $(f^{-1} \cdot g^{-1})'(2)$

6. Let $f(x) = x^2e^{x^2}$, $x \in \mathbb{R}$.

   6.1 Show that f^{-1} exists and its differentiable on $(0, \infty)$.

   6.2 Compute $(f^{-1})'(e)$

7. Let $f(x) = x + e^{2x}$ where $x \in \mathbb{R}$.

   7.1 Show that f is 1-1 on $x \in \mathbb{R}$.

   7.2 Use the result from 7.1 and the IFT to explain that f differentiable on $\mathbb{R}$.

   7.3 Compute $(f^{-1})'(4 + \ln 2)$.

8. Use the Inverse Function Theorem, prove that

   8.1 $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$ where $x \in (-1, 1)$

   8.2 $(\arctan x)' = \dfrac{1}{1+x^2}$ where $x \in (-\infty, \infty)$

8.3 $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ where $x \in (0, \infty)$

9. Use the IFT to find derivative of invrese function $f(x) = e^x - e^{-x}$ where $x \in \mathbb{R}$.

10. Suppose that f' exists and continuous on a nonempty, open interval (a, b) with $f'(x) \neq 0$ for all $x \in (a, b)$.

    10.1 Prove that f is 1-1 on (a, b) and takes (a, b) onto some open interval (c, d)

    10.2 Show that $(f^{-1})'$ exists and continuous on (c, d)

    10.3 Use the function $f(x) = x^3$, show that 7.2 is false if the assumption $f'(x) \neq 0$ fails to hold for some $x \in (c, d)$

11. Let $[a, b]$ be a closed, bounded interval. Find all functions f that satisfy the following conditions for some fixed $\alpha > 0$: f is continuous and 1-1 on $[a, b]$,

$$f'(x) \neq 0 \text{ and } f'(x) = \alpha(f^{-1})'(f(x)) \text{ for all } x \in (a, b).$$

12. Let f be differentiable at every point in a closed, bounded interval $[a, b]$. Prove that if f' is increasing on (a, b), then f' is continuous on (a, b).

13. Suppose that f is increasing on $[a, b]$. Prove that

    13.1 if $x_0 \in [a, b)$, then $f(x_0^+)$ exists and $f(x_0) \leq f(x_0^+)$,

    13.2 if $x_0 \in (a, b]$, then $f(x_0^-)$ exists and $f(x_0^-) \leq f(x_0)$.

Chapter 7

Integrability on $\mathbb{R}$

7.1 Riemann integral

PARTITION.

Definition 7.1.1 *Let $a, b \in \mathbb{R}$ with $a < b$.*

1. *A **partition** of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, , ..., x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

2. *The **norm** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is the number*

$$\|P\| = \max_{1 \leq j \leq n} |x_j - x_{j-1}|.$$

3. *A **refinement** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is a partition Q of $[a, b]$ that satisfies $Q \supseteq P$. In this case we say that Q is **finer** than P or Q is a **refinement** of P.*

Example 7.1.2 *Give example of partition and refinement of the interval $[0, 1]$.*

Partitions	Norms of Partition
$P = \{0, 0.5, 1\}$	$\|P\| = 0.5$
$Q = \{0, 0.25, 0.5, 0.75, 1\}$	$\|Q\| = 0.25$
$R = \{0, 0.2, 0.3, , 0.5, 0.6, 0.8, 1\}$	$\|R\| = 0.2$

We see that Q and R are refinements of P but R is not a refinement of Q.

Example 7.1.3 *Prove that for each $n \in \mathbb{N}$,*

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

is a partition of the interval $[0, 1]$ and find a norm of P_n.

Solution. Let $n \in \mathbb{N}$. It is easy to see that

$$0 = \frac{0}{n} < \frac{1}{n} < \frac{2}{n} < ... < \frac{n}{n} = 1.$$

Thus, P_n is a partition of $[0, 1]$. We have

$$\|P_n\| = \max_{1 \le j \le n} \left| \frac{j}{n} - \frac{j-1}{n} \right| = \frac{1}{n}.$$

Example 7.1.4 (**Dyadic Partition**) *Let $n \in \mathbb{N}$ and define*

$$P_n = \left\{ \frac{j}{2^n} : j = 0, 1, ..., 2^n \right\}.$$

1. *Prove that P_n is a partition of the interval $[0, 1]$.*

2. *Prove that P_m is finer than P_n when $m > n$.*

3. *Find a norm of P_n.*

Solution. Let $n \in \mathbb{N}$. It is easy to see that

$$0 = \frac{0}{2^n} < \frac{1}{2^n} < \frac{2}{2^n} < ... < \frac{2^n}{2^n} = 1.$$

Thus, P_n is a partition of $[0, 1]$. Next, we will show that $P_n \subseteq P_m$ if $m > n$. Let $m > n$ and $x \in P_n$. Then there is a $j \in \{0, 1, 2, ..., 2^n\}$ such that $x = \dfrac{j}{2^n}$. Since $m > n$, $m - n > 0$. Then $2^{m-n} > 0$. From $0 \le j \le 2^n$, it implies that

$$0 \le j \cdot 2^{m-n} \le 2^n \cdot 2^{m-n} = 2^m.$$

We obtain

$$x = \frac{j \cdot 2^m}{2^n \cdot 2^m} = \frac{j \cdot 2^{m-n}}{2^m} \in P_m.$$

Thus, P_m is finer than P_n when $m > n$. We final have

$$\|P_n\| = \max_{1 \le j \le n} \left| \frac{j}{2^n} - \frac{j-1}{2^n} \right| = \frac{1}{2^n}.$$

UPPER AND LOWER RIEMANN SUM.

Definition 7.1.5 *Let $a, b \in \mathbb{R}$ with $a < b$, let $P = \{x_0, x_1, , ..., x_n\}$ be a partition of the interval $[a, b]$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded.*

1. *The* ***upper Riemann sum*** *of f over P is the number*

$$U(f, P) := \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1})$$

where

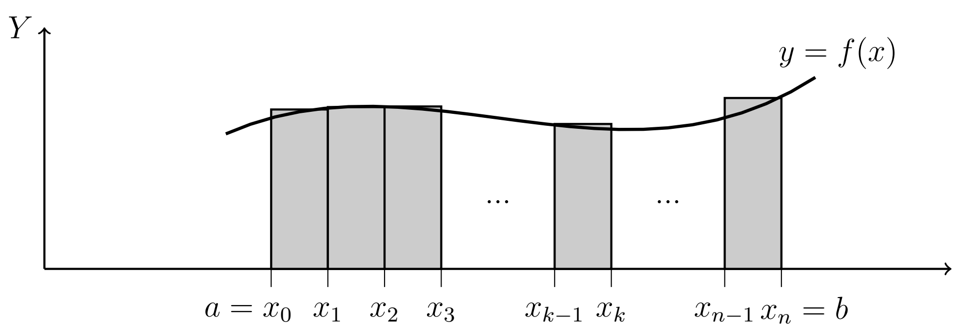
$$M_j(f) := \sup_{x \in [x_{j-1}, x_j]} f(x).$$

2. *The* ***lower Riemann sum*** *of f over P is the number*

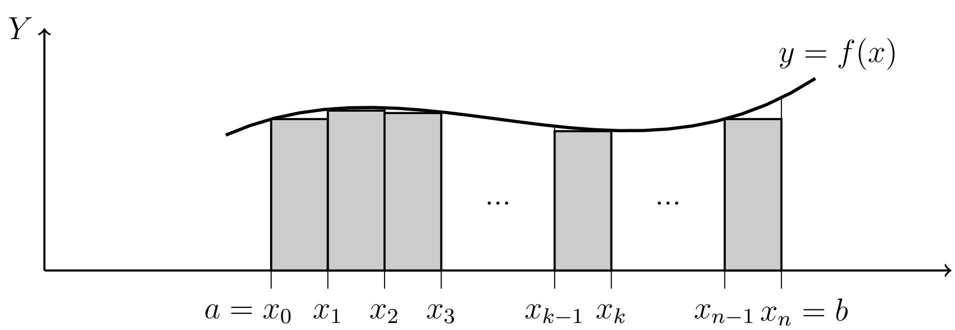
$$L(f, P) := \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1})$$

where

$$m_j(f) := \inf_{x \in [x_{j-1}, x_j]} f(x).$$



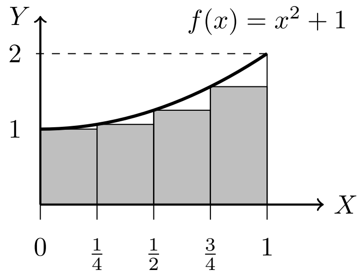
Upper Riemann Sum

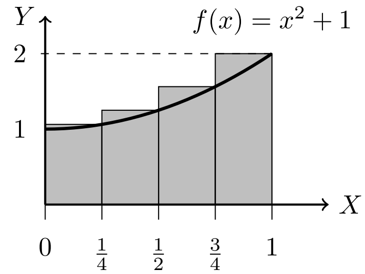


Lower Riemann Sum

Example 7.1.6 *Let $f(x) = x^2 + 1$ where $x \in [0,1]$. Find $L(f,P)$ and $U(f,P)$*

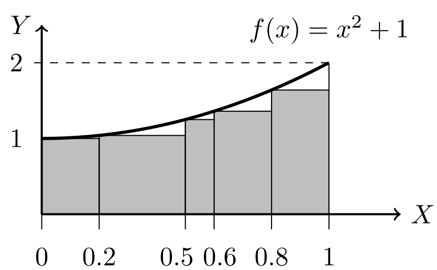
1. $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$

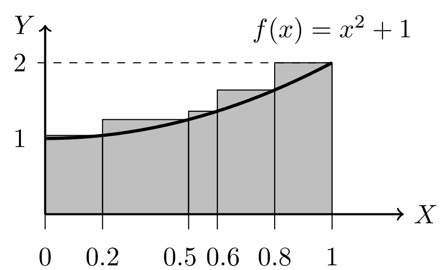




$$\begin{aligned}
L(P,f) &= \frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) \\
&= \frac{1}{4}\left(1 + \frac{17}{16} + \frac{5}{4} + \frac{25}{16}\right) = \frac{79}{64} \\
U(P,f) &= \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) + \frac{1}{4}f(1) \\
&= \frac{1}{4}\left(\frac{17}{16} + \frac{5}{4} + \frac{25}{16} + 2\right) = \frac{47}{32}
\end{aligned}$$

2. $P = \{0, 0.2, 0.5, 0.6, 0.8, 1\}$



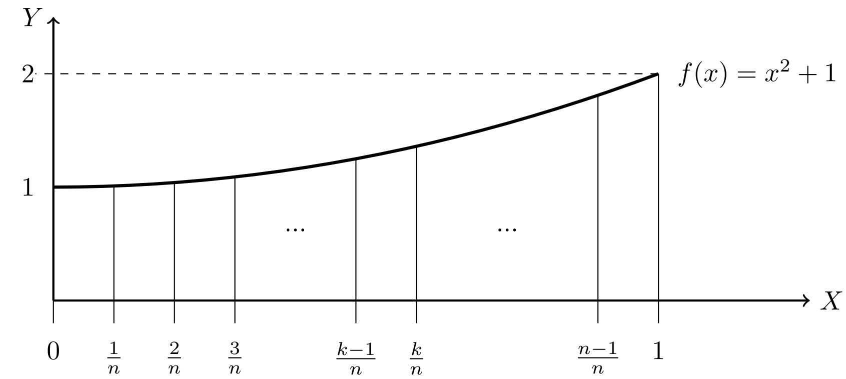


$$\begin{aligned}
L(P,f) &= 0.2f(0) + 0.3f(0.2) + 0.1f(0.5) + 0.2f(0.6) + 0.2f(0.8) \\
&= 0.2(1) + 0.3(1.04) + 0.1(1.25) + 0.2(1.36) + 0.2(1.64) \\
&= 1.237 \\
U(P,f) &= 0.2f(0.2) + 0.3f(0.5) + 0.1f(0.6) + 0.2f(0.8) + 0.2f(1) \\
&= 0.2(1.04) + 0.3(1.25) + 0.1(1.36) + 0.2(1.64) + 0.2(2) \\
&= 1.447
\end{aligned}$$

Example 7.1.7 *Let $f(x) = x^2 + 1$ where $x \in [0, 1]$. Find $L(P_n, f)$ and $U(P_n, f)$ for $n \in \mathbb{N}$ if*

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}.$$

Solution. Let $x_k = \dfrac{k}{n}$ and $\Delta x_k = x_k - x_{k-1} = \dfrac{1}{n}$ for each $k = 0, 1, 2, ..., n$.



For interval $[x_{k-1}, x_k]$ and f is increasing on $[0, 1]$, it follows that

$$m_k = f(x_{k-1}) = f\left(\frac{k-1}{n}\right) = \left(\frac{k-1}{n}\right)^2 + 1 = \frac{1}{n^2}(k-1)^2 + 1$$
$$m_k = f(x_k) = f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2 + 1 = \frac{1}{n^2} \cdot k^2 + 1$$

Thus, we obtain

$$\begin{aligned} L(P_n, f) &= \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \left[\frac{1}{n^2}(k-1)^2 + 1\right] \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 + \frac{1}{n} \sum_{k=1}^{n} 1 \\ &= \frac{1}{n^3}[0^2 + 1^2 + 2^2 + \cdots + (n-1)^2] + \frac{1}{n} \cdot n \\ &= \frac{1}{n^3} \cdot \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} + 1 \\ &= \frac{(n-1)(n)(2n-1)}{6n^3} + 1 = \frac{(n-1)(2n-1)}{6n^2} + 1 \end{aligned}$$

and

$$\begin{aligned} U(P_n, f) &= \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} \left[\frac{1}{n^2} \cdot k^2 + 1\right] \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 + \frac{1}{n} \sum_{k=1}^{n} 1 \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n} \cdot n \\ &= \frac{(n+1)(2n+1)}{6n^2} + 1. \end{aligned}$$

Theorem 7.1.8 *$L(f,P) \le U(f,P)$ for all partition P and all bounded function f.*

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition and f be bounded on $[a,b]$. Then

$$m_j(f) = \inf_{x\in[x_{j-1},x_j]} f(x) \le \sup_{x\in[x_{j-1},x_j]} f(x) = M_j(f) \quad \text{for all } j = 1,2,..,n.$$

It follows that

$$L(f,P) = \sum_{j=1}^{n} m_j(f)\Delta x_j \le \sum_{j=1}^{n} M_j(f)\Delta x_j = U(f,P).$$

□

Theorem 7.1.9 (**Sum Telescopes**) *If $g : \mathbb{N} \to \mathbb{R}$, then*

$$\sum_{k=m}^{n} [g(k+1) - g(k)] = g(n+1) - g(m)$$

for all $n \ge m$ in $\mathbb{N}$.

Proof. Fix $m \in \mathbb{N}$. We will prove by induction on n. The Sum Telescopes is obvious for $n = 1$. Assume that the Sum Telescopes is true for some $n \in \mathbb{N}$. By inductive hypothesis,

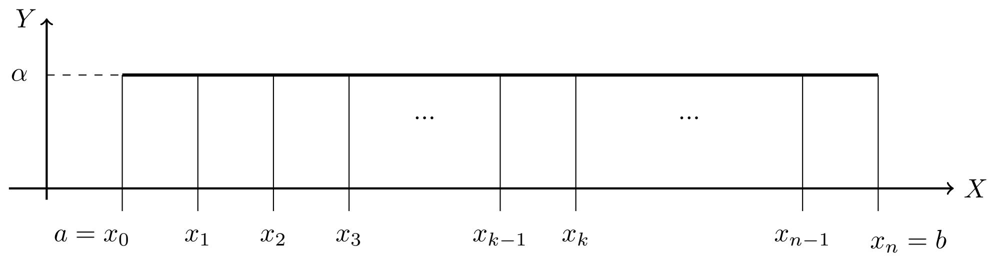
$$\begin{aligned}\sum_{k=m}^{n+1} [g(k+1) - g(k)] &= \sum_{k=m}^{n} [g(k+1) - g(k)] + [g(n+2) - g(n+1)] \\ &= g(n+1) - g(m) + [g(n+2) - g(n+1)] \\ &= g(n+2) - g(m).\end{aligned}$$

The Sum Telescopes is true for some $n+1$. We conclude that by induction that the Sum Telescopes holds for $n \in \mathbb{N}$. □

Theorem 7.1.10 *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$U(f, P) = L(f, P) = \alpha(b - a)$$

Proof. Let $f(x) = \alpha$ is constant on $[a, b]$ and let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[a, b]$ such that $x_0 = a$ and $x_n = b$.



For each $j \in \{1, 2, ..., n\}$ and $f(x) = \alpha$, we have

$$m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x) = \alpha \quad \text{and} \quad M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) = \alpha.$$

Use the Sum Telescopes, we obtain

$$L(P, f) = \sum_{j=1}^{n} m_j(f)\Delta x_j = \sum_{j=1}^{n} \alpha(x_j - x_{j-1}) = \alpha(x_n - x_0) = \alpha(b - a),$$
$$U(P, f) = \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} \alpha(x_j - x_{j-1}) = \alpha(x_n - x_0) = \alpha(b - a).$$

□

Theorem 7.1.11 *If P is any partition of $[a, b]$ and Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. It is clear that $L(f, Q) \leq U(f, Q)$ by Theorem 7.1.8.

Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[a, b]$ such that $x_0 = a$ and $x_n = b$.

Assume that Q is a refinement of P. Special case $Q = P \cup \{c\}$ for some $c \in (a, b)$.

If $c \in P$, then $Q = P$ which implies that

$$L(f, P) = L(f, Q) \leq U(f, Q) = U(f, P).$$

The proof is done for this case.

Suppose $c \notin P$. Then there is an x_k such that

$$x_{k-1} < c < x_k \quad \text{for some } k \in \{1, 2, ..., n\}.$$

Consider

$$U(f,P) = \sum_{j=1}^{k-1} M_j(f)\Delta x_j + \sup_{x\in[x_{k-1},x_k]} f(x)\cdot(x_k - x_{k-1}) + \sum_{j=K+1}^{n} M_j(f)\Delta x_j$$
$$U(f,Q) = \sum_{j=1}^{k-1} M_j(f)\Delta x_j + \sup_{x\in[x_{k-1},c]} f(x)\cdot(c - x_{k-1}) + \sup_{x\in[c,x_k]} f(x)\cdot(x_k - c) + \sum_{j=k+1}^{n} M_j(f)\Delta x_j$$

Set $M = \sup_{x\in[x_{k-1},x_k]} f(x)$. Then

$$\sup_{x\in[x_{k-1},c]} f(x) \le M \quad \text{and} \quad \sup_{x\in[c,x_k,c]} f(x) \le M.$$

We obtain

$$\begin{aligned} U(f,P) - U(f,Q) &= \sup_{x\in[x_{k-1},x_k]} f(x)\cdot(x_k - x_{k-1}) - \sup_{x\in[x_{k-1},c]} f(x)\cdot(c - x_{k-1}) - \sup_{x\in[c,x_k]} f(x)\cdot(x_k - c) \\ &\ge M(x_k - x_{k-1}) - M(c - x_{k-1}) - M(x_k - c) \\ &= M(x_k - x_{k-1} - c + x_{k-1} - x_k + c) = 0. \end{aligned}$$

Thus, $U(f,P) \ge U(f,Q)$. A similar argument show that $L(f,P) \le L(f,Q)$. □

Corollary 7.1.12 *If P and Q are any partitions of $[a,b]$, then*

$$L(f,P) \le U(f,Q).$$

Proof. Assume that P and Q are any partitions of $[a,b]$. Then

$$P \subseteq P \cup Q \quad \text{and} \quad Q \subseteq P \cup Q.$$

Thus, $P \cup Q$ is a refinement of P and Q. By Theorem 7.1.11, it implies that

$$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,P).$$

Hence, $L(f,P) \le U(f,Q)$. □

RIEMANN INTEGRABLE.

Definition 7.1.13 *Let $a, b \in \mathbb{R}$ with $a < b$.*
*A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** or **integrable** on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 7.1.14 *Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Assume that f is continuous on the interval $[a, b]$.
It follows that f is bounded on $[a, b]$ by the EVT. Theorem 5.3.6 implies that f is uniformly continuous on the interval $[a, b]$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that

$$|x - y| < \delta \text{ and } x, y \in [a, b] \quad \text{imply} \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}. \tag{7.1}$$

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$ such that $\|P\| < \delta$.
Fix $j \in \{1, 2, ..., n\}$. By agian the EVT, there are $x_n, x_M \in [x_{j-1}, x_j]$ such that

$$f(x_m) = m_j(f) \quad \text{and} \quad f(x_M) = M_j(f).$$

Since $\|P\| < \delta$, we have $|x_M - x_m| \le |x_j - x_{j-1}| < \delta$. Then x_m, x_M satisfy (7.1), it implies that

$$|M_j(f) - m_j(f)| = |f(x_M) - f(x_m)| < \frac{\varepsilon}{b - a}.$$

Use the Sum Telescopes, We obtain

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=1}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) \\ &< \sum_{j=1}^{n} \frac{\varepsilon}{b - a} \cdot (x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b - a} \cdot (x_n - x_0) = \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon. \end{aligned}$$

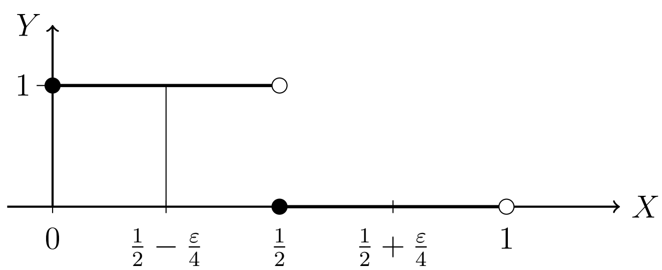
Therefore, f is integrable on $[a, b]$. □

Example 7.1.15 *Prove that the function*

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

is integrable on $[0, 1]$.

Solution. Let $\varepsilon > 0$. Case $\varepsilon < 1$. Choose $P = \left\{0, \dfrac{1}{2} - \dfrac{\varepsilon}{4}, \dfrac{1}{2} + \dfrac{\varepsilon}{4}, 1\right\}$.



We obtain

$$U(f,P) = 1\left[\left(\frac{1}{2} - \frac{\varepsilon}{4}\right) - 0\right] + 1\left[\left(\frac{1}{2} + \frac{\varepsilon}{4}\right) - \left(\frac{1}{2} - \frac{\varepsilon}{4}\right)\right] + 0\left[1 - \left(\frac{1}{2} + \frac{\varepsilon}{4}\right)\right] = \frac{1}{2} + \frac{\varepsilon}{4}$$
$$L(f,P) = 1\left[\left(\frac{1}{2} - \frac{\varepsilon}{4}\right) - 0\right] + 0\left[\left(\frac{1}{2} + \frac{\varepsilon}{4}\right) - \left(\frac{1}{2} - \frac{\varepsilon}{4}\right)\right] + 0\left[1 - \left(\frac{1}{2} + \frac{\varepsilon}{4}\right)\right] = \frac{1}{2} - \frac{\varepsilon}{4}$$
$$U(f,P) - L(f,P) = \frac{\varepsilon}{2} < \varepsilon.$$

Case $\varepsilon \ge 1$. Choose $P = \left\{0, \dfrac{1}{2}, 1\right\}$. Then

$$U(f,P) = 1\left(\frac{1}{2} - 0\right) + 0\left(1 - \frac{1}{2}\right) = \frac{1}{2}$$
$$L(f,P) = 0\left(\frac{1}{2} - 0\right) + 0\left(1 - \frac{1}{2}\right) = 0$$
$$U(f,P) - L(f,P) = \frac{1}{2} < 1 \le \varepsilon.$$

Thus, f is integrable on $[0, 1]$.

Example 7.1.16 (**Dirichlet function**) *Prove that the function*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is NOT Riemann integrable on $[0, 1]$.

Solution. Suppose that f is Riemann integrable on $[0, 1]$.

Given $\varepsilon = \frac{1}{2}$. There is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[0, 1]$ such that

$$U(f, P) - L(f, P) < \frac{1}{2}.$$

Fix $j \in \{1, 2, ..., n\}$. By real property, it leads to that there are $r \in Q$ and $s \in \mathbb{Q}^c$ such that $r, s \in [x_{j-1}, x_j]$. It implies that

$$m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x) = 0 \quad \text{and} \quad M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) = 1.$$

Use the Sum Telescopes, we obtain

$$U(f, P) = \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = x_n - x_0 = 1 - 0 = 1$$

$$L(f, P) = \sum_{j=1}^{n} m_j(f)\Delta x_j = \sum_{j=1}^{n} 0(x_j - x_{j-1}) = 0$$

$$U(f, P) - L(f, P) = 1 - 0 = 1 > \frac{1}{2},$$

a contradiction. We conclude that the Dirichlet function is not Riemann integrable on $[0, 1]$.

UPPER AND LOWER INTEGRABLE.

Definition 7.1.17 *Let* $a, b \in \mathbb{R}$ *with* $a < b$, *and* $f : [a, b] \to \mathbb{R}$ *be bounded.*

1. *The* ***upper integral*** *of* f *on* $[a, b]$ *is the number*

$$(U) \int_a^b f(x)\, dx := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

2. *The* ***lower integral*** *of* f *on* $[a, b]$ *is the number*

$$(L) \int_a^b f(x)\, dx := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

3. *If the upper and lower integrals of* f *on* $[a, b]$ *are equal, we define the* ***integral*** *of* f *on* $[a, b]$ *to be the common value*

$$\int_a^b f(x)\, dx := (U) \int_a^b f(x)\, dx = (L) \int_a^b f(x)\, dx.$$

Example 7.1.18 *Let $f(x) = \alpha$ where $x \in [a, b]$. Show that*

$$(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx = \alpha(b-a).$$

Solution. By Theorem 7.1.10, for any partition of $[a, b]$, we have $U(f, P) = L(f, P) = \alpha(b - a)$. It follows that

$$(U)\int_a^b f(x)\,dx = \inf_P U(f,P) = \alpha(b-a),$$
$$(L)\int_a^b f(x)\,dx = \sup_P L(f,P) = \alpha(b-a).$$

Example 7.1.19 *The Dirichlet function is defined*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Find the upper integral and lower integral of the Dirichlet function on $[0, 1]$.

Solution. By Example 7.1.16, for any partition of $[a, b]$, we have $U(f, P) = 1$ and $L(f, P) = 0$. It follows that

$$(U)\int_a^b f(x)\,dx = \inf_P U(f,P) = 1,$$
$$(L)\int_a^b f(x)\,dx = \sup_P L(f,P) = 0.$$

Theorem 7.1.20 *If $f : [a, b] \to \mathbb{R}$ is bounded, then its upper and lower integrals exist and are finite, and satisfy*

$$(L)\int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

Proof. By Corollary 7.1.12, we have

$$L(f, P) \le U(f, Q) \text{ for partitions } P, Q \text{ of } [a, b].$$

We obtain by taking supremum over all partitions P of $[a, b]$,

$$(L)\int_a^b f(x)\,dx = \sup_P L(f,P) \le \sup_P U(f,Q) = U(f,Q).$$

Taking infimum over all partitions Q of $[a, b]$, we have

$$(L)\int_a^b f(x)\,dx \le \inf_Q U(f,Q) = (U)\int_a^b f(x)\,dx.$$

Hence, $(L)\int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx$. □

Theorem 7.1.21 *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if*

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx.$$

Proof. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded.

Assume that f is integrable on $[a, b]$. Let $\varepsilon > 0$. There is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

By definition,

$$L(f, P) \le (L)\int_a^b f(x)\,dx \quad \text{and} \quad (U)\int_a^b f(x)\,dx \le U(f, P).$$

By Theorem 7.1.20, it follows that

$$\begin{aligned}\left|(U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx\right| &= (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \\ &\le U(f, P) - L(f, P) < \varepsilon.\end{aligned}$$

Thus, $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$.

Conversely, we assume that $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$.

Let $\varepsilon > 0$. Choose, by the API and APS, partitions P_1, P_2 of $[a, b]$ such that

$$(L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f, P_1) \quad \text{and} \quad U(f, P_2) < (U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Then P is a refinement of P_1 and P_2. By Theorem 7.1.11, it follows that

$$\begin{aligned}U(P, f) - L(f, P) &\le U(f, P_2) - L(f, P_1) \\ &< \left((U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}\right) - \left((L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}\right) = \varepsilon.\end{aligned}$$

Therefore, f is integrable on $[a, b]$. □

Theorem 7.1.22 *For a constant* α,

$$\int_a^b \alpha\, dx = \alpha(b-a).$$

Proof. It is easy to prove by Example 7.1.18 and Theorem 7.1.21. □

Example 7.1.23 *Let* $f:[0,2]\to\mathbb{R}$ *defined by*

$$f(x)=\begin{cases}0 & \textit{if } x\neq 1\\ 3 & \textit{if } x=1\end{cases}$$

Show that f *is integrable and find* $\displaystyle\int_0^2 f(x)dx$.

Solution. Let $\varepsilon>0$. Let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[0,2]$ such that $\|P\|<\dfrac{\varepsilon}{6}$. Then

$$m_j(f)=\inf_{x\in[x_{j-1},x_j]} f(x)=0 \quad \text{for all } j=1,2,...,n.$$

We obtain $L(f,P)=\displaystyle\sum_{j=1}^n m_j(f)\Delta x_j=0$ which is not depend on ε. So,

$$(L)\int_0^2 f(x)\,dx=\sup_P L(f,P)=0.$$

Case $1\in P$. Then $x_k=1$ for some $k\in\{1,2,...,n-1\}$. We have

$$M_j(f)=\inf_{x\in[x_{j-1},x_j]} f(x)=0 \text{ for all } j\neq k,k+1 \quad \text{and} \quad M_k(f)=3,\ M_{k+1}(f)=3.$$

From $\|P\|<\dfrac{\varepsilon}{6}$, it follows that $|x_j-x_{j-1}|<\dfrac{\varepsilon}{6}$ for all $j=1,2,...,n$. We obtain

$$\begin{aligned}U(f,P)-L(f,P)&=U(f,P)-0\\&=\sum_{j=1}^n M_j(f)\Delta x_j=3(x_k-x_{k-1})+3(x_{k+1}-x_k)<3\cdot\frac{\varepsilon}{6}+3\cdot\frac{\varepsilon}{6}=\varepsilon.\end{aligned}$$

Case $1\notin P$. Then $1\in[x_{k-1},x_k]$ for some $k\in\{1,2,...,n\}$. We have

$$M_j(f)=\inf_{x\in[x_{j-1},x_j]} f(x)=0 \text{ for all } j\neq k \quad \text{and} \quad M_k(f)=3.$$

We obtain

$$U(f,P)-L(f,P)=U(f,P)-0=\sum_{j=1}^n M_j(f)\Delta x_j=3(x_k-x_{k-1})<3\cdot\frac{\varepsilon}{6}=\frac{\varepsilon}{2}<\varepsilon.$$

Thus, f is integrable on $[0,2]$ and

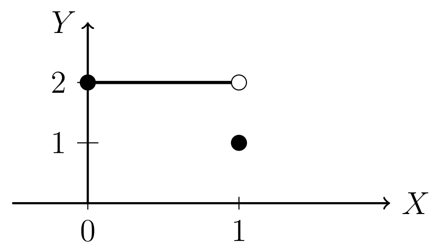
$$\int_0^2 f(x)\,dx=(L)\int_0^2 f(x)\,dx=0.$$

Example 7.1.24 *Let $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 2 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Show that f is integrable and find $\int_0^1 f(x)dx$.

Solution. Let $\varepsilon > 0$. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0,1]$ such that $\|P\| < \varepsilon$.



Then, $M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) = 2 \quad$ for all $j = 1, 2, ..., n$.
We obtain

$$U(f,P) = \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} 2(x_j - x_{j-1}) = 2(x_n - x_0) = 2(1-0) = 2$$

which is not depend on ε. So,

$$(U)\int_0^2 f(x)\,dx = \inf_P U(f,P) = 2.$$

We see that

$$m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x) = 2 \text{ for all } j \neq n \quad \text{and} \quad M_n(f) = 1.$$

From $\|P\| < \varepsilon$, it follows that $|x_j - x_{j-1}| < \varepsilon$ for all $j = 1, 2, ..., n$. We obtain

$$\begin{aligned} U(f,P) - L(f,P) &= 2 - L(f,P) \\ &= 2 - \sum_{j=1}^{n} m_j(f)\Delta x_j = 2 - \sum_{j=1}^{n-1} 2(x_j - x_{j-1}) - 1(x_n - x_{n-1}) \\ &= 2 - 2(x_{n-1} - x_0) - 1(x_n - x_{n-1}) \\ &= 2 - 2(x_{n-1} - 0) - 1(1 - x_{n-1}) = 1 - x_{n-1} = x_n - x_{n-1} < \varepsilon. \end{aligned}$$

Thus, f is integrable on $[0,1]$ and

$$\int_0^1 f(x)\,dx = (U)\int_0^2 f(x)\,dx = 2.$$

Exercises 7.1

1. For each of the following, compute $U(f,P)$, $L(f,P)$, and $\int_0^1 f(x)\,dx$, where

$$P = \left\{0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1\right\}.$$

   Find out whether the lower sum or the upper sum is better approximation to the integral. Graph f and explain why this is so.

   1.1 $f(x) = 1 - x^2$ 1.2 $f(x) = 2x^2 + 1$ 1.3 $f(x) = x^2 - x$

2. Let $P_n = \left\{\frac{j}{n} : n = 0, 1, ..., n\right\}$ for each $n \in \mathbb{N}$. Prove that a bounded function f is integrable on $[0,1]$ if

$$I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

   in which case $\int_0^1 f(x)\,dx$ equals I_0.

3. For each of the following functions, use P_n in 2. to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)\,dx$.

   3.1 $f(x) = x$

   3.2 $f(x) = x^2$

   3.3 $f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

4. Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that the function $f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if otherwise} \end{cases}$ is integrable on $[0,1]$. What is the value of $\int_0^1 f(x)\,dx$?

5. Suppose that f is continuous on an interval $[a,b]$. Show that $\int_a^c f(x)\,dx = 0$ for all $c \in [a,b]$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

6. Let f be bounded on a nondegenerate interval $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that

$$P \supseteq P_\varepsilon \quad \text{imples} \quad |U(f,P) - L(f,P)| < \varepsilon.$$

7.2 Riemann sums

Definition 7.2.1 *Let* $f : [a, b] \to \mathbb{R}$.

1. *A* ***Riemann sum*** *of* f *with respect to a partition* $P = \{x_0, x_1, ..., x_n\}$ *of* $[a, b]$ *is a sum of the form*
$$\sum_{j=1}^{n} f(t_j)\Delta x_j,$$
where the choice of $t_j \in [x_{j-1}, x_j]$ *is arbitrary.*

2. *The Riemann sums of* f *are* ***converge*** *to* $I(f)$ *as* $\|P\| \to 0$ *if and only if given* $\varepsilon > 0$ *there is a partition* P_ε *of* $[a, b]$ *such that*
$$P = \{x_0, x_1, ..., x_n\} \supseteq P_\varepsilon \quad \textit{implies} \quad \left| \sum_{j=1}^{n} f(t_j)\Delta x_j - I(f) \right| < \varepsilon$$
for all choice of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$. *In this case we shall use the notation*
$$I(f) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j.$$

Example 7.2.2 *Let* $f(x) = x^2$ *where* $x \in [0, 1]$ *and* $P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$ *be a partition of* $[0, 1]$. *Show that if* $f(t_i)$ *is choosen by the right end point and left end point in each subinterval, then two* $I(f)$, *depend on two methods, are NOT different.*

Solution. The Right End Point : Choose $f(t_j) = f(\frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ and have $\Delta x_j = \dfrac{j}{n} - \dfrac{j-1}{n} = \dfrac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain
$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2 = \frac{1}{n^3}\sum_{j=1}^{n} j^2 \\ &= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

Thus,
$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3} = \frac{1}{3}.$$

The Left End Point : Choose $f(t_j) = f(\frac{j-1}{n})$ on the subinterval $[x_{j-1}, x_j]$. We obtain

$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j-1}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-1}{n}\right)^2 = \frac{1}{n^3}\sum_{j=1}^{n}(j-1)^2 \\ &= \frac{1}{n^3}\left[0^2+1^2+2^2+\cdots(n-1)^2\right] \\ &= \frac{1}{n^3}\cdot\frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)}{6n^2}.\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}.$$

Theorem 7.2.3 *Let $a, b \in \mathbb{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable on $[a, b]$ if and only if*

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j$$

exists, in which case

$$I(f) = \int_a^b f(x)\,dx.$$

Proof. Assume that f is Riemann integrable on $[a, b]$.

Let $\varepsilon > 0$. By the API and APS, there is a partition P_ε of $[a, b]$ such that

$$\int_a^b f(x)\,dx + \varepsilon < L(f, P_\varepsilon) \quad \text{and} \quad U(f, P_\varepsilon) < (U)\int_a^b f(x)\,dx + \varepsilon.$$

Let $P = \{x_0, x_1, ..., x_n\} \supseteq P_\varepsilon$. From $m_j(f) \le f(t_j) \le M_j(f)$ for any choice of $t_j \in [x_{j-1}, x_j]$. Hence,

$$\begin{aligned}\int_a^b f(x)\,dx - \varepsilon < L(f, P_\varepsilon) < L(f, P) &\le \sum_{j=1}^{n} f(t_j)\Delta x_j \\ &\le U(f, P) < U(f, P_\varepsilon) < \int_a^b f(x)\,dx + \varepsilon.\end{aligned}$$

It implies that

$$\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| < \varepsilon.$$

for all partitions $P \supseteq P_\varepsilon$ and all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$.

Conversely, assume that the Riemann sums of converge to $I(f)$. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that

$$\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \frac{\varepsilon}{3} \tag{7.2}$$

for all choices of $t_j \in [x_{j-1}, x_j]$. By the API and APS, choose $u_j, v_j \in [x_{j-1}, x_j]$ such that

$$M_j(f) - \frac{\varepsilon}{6(b-a)} < f(u_j) \quad \text{and} \quad f(v_j) < m_j(f) + \frac{\varepsilon}{6(b-a)}$$

It implies that

$$f(u_j) - f(v_j) > M_j(f) - \frac{\varepsilon}{6(b-a)} - m_j(f) - \frac{\varepsilon}{6(b-a)} = M_j(f) - m_j(f) - \frac{\varepsilon}{3(b-a)}.$$

So,

$$M_j(f) - m_j(f) < f(u_j) - f(v_j) + \frac{\varepsilon}{3(b-a)}.$$

By (7.2) and telescoping, we have

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{j=1}^{n}(M_j(f) - m_j(f))\Delta x_j \\
&< \sum_{j=1}^{n} f(u_j)\Delta x_j - \sum_{j=1}^{n} f(v_j)\Delta x_j + \frac{\varepsilon}{3(b-a)}\sum_{j=1}^{n}(x_j - x_{j-1}) \\
&\le \left|\sum_{j=1}^{n} f(u_j)\Delta x_j - \sum_{j=1}^{n} f(v_j)\Delta x_j\right| + \frac{\varepsilon}{3(b-a)}(x_n - x_0) \\
&= \left|\sum_{j=1}^{n} f(u_j)\Delta x_j - I(f) - \sum_{j=1}^{n} f(v_j)\Delta x_j + I(f)\right| + \frac{\varepsilon}{3(b-a)}(b-a) \\
&\le \left|\sum_{j=1}^{n} f(u_j)\Delta x_j - I(f)\right| + \left|\sum_{j=1}^{n} f(v_j)\Delta x_j - I(f)\right| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, f is Riemann integrable on $[a, b]$. □

Theorem 7.2.4 (**Linear Property**) *If f, g are integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then $f+g$ and αf are integrable on $[a,b]$. In fact,*

1. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

Proof. Assume that f and g are integrable on $[a,b]$ and $\alpha \in \mathbb{R}$.

Let $\varepsilon > 0$ and choose P_ε such that for any partition $P = \{x_0, x_1, ..., x_n\} \supseteq P_\varepsilon$ of $[a,b]$ and any choice of $t_j \in [x_{j-1}, x_j]$, we have

$$\left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\sum_{j=1}^n g(t_j)\Delta x_j - \int_a^b g(x)\,dx\right| < \frac{\varepsilon}{2}.$$

By triangle inequality, for any choice $t_j \in [x_{j-1}, x_j]$,

$$\begin{aligned}\left|\sum_{j=1}^n (f+g)(t_j)\Delta x_j - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| &= \left|\sum_{j=1}^n f(t_j)\Delta x_j + \sum_{j=1}^n g(t_j)\Delta x_j - \int_a^b f(x)\,dx - \int_a^b f(x)\,dx\right| \\ &\leq \left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| + \left|\sum_{j=1}^n g(t_j)\Delta x_j - \int_a^b g(x)\,dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We conclude that $f+g$ is integrable on $[a,b]$ and $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Similarly, if P_ε is chosen so that if $P = \{x_0, x_1, ..., x_n\}$ is finer than P_ε, then

$$\left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{|\alpha|+1}.$$

It is easy to see that, for any choice $t_j \in [x_{j-1}, x_j]$,

$$\begin{aligned}\left|\sum_{j=1}^n \alpha f(t_j)\Delta x_j - \alpha\int_a^b f(x)\,dx\right| &= |\alpha|\left|\sum_{j=1}^n f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| \\ &< |\alpha| \cdot \frac{\varepsilon}{|\alpha|+1} < \varepsilon.\end{aligned}$$

Thus, αf is integrable on $[a,b]$ and $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx$. □

Theorem 7.2.5 *If f is integrable on $[a, b]$, then f is integrable on each subinterval $[c, d]$ of $[a, b]$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for all $c \in (a, b)$.

Proof. We may suppose that $a < b$. Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Let $P_0 = P \cup \{c\}$ and $P_1 = P_0 \cap [a, c]$. Since P_1 is a partition of $[a, c]$ and P_0 is a refinement of P, we have

$$U(f, P_1) - L(f, P_1) \le U(f, P_0) - L(f, P_0) \le U(f, P) - L(f, P) < \varepsilon.$$

Therefore, f is integrable on $[a, c]$. A similar argument proves that f is integrable on any subinterval $[c, d]$ of $[a, b]$.

Let $P_2 = P_0 \cap [c, d]$. Then $P_0 = P_1 \cup P_2$ and by definition

$$\begin{aligned} U(f, P) &\ge U(f, P_0) = U(f, P_1) + U(f, P_2) \\ &\ge (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \end{aligned}$$

Next, we will take infimum of the last inequality over all partitions P of $[a, b]$, we obtain

$$\begin{aligned} \int_a^b f(x)\,dx &= (U)\int_a^b f(x)\,dx \\ &= \inf_P U(f, P) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \end{aligned}$$

A similar argument using lower integrals shows that

$$\int_a^b f(x)\,dx \le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

We conclude that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. □

By Theorem 7.2.5, we obtain

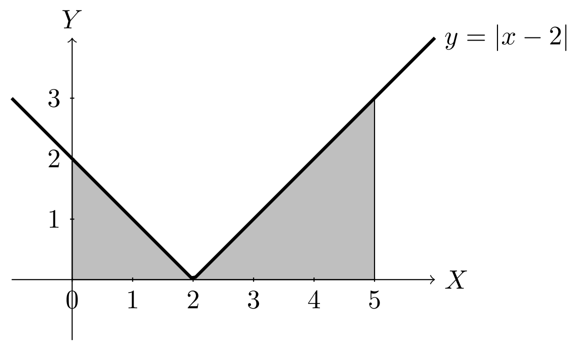
$$\int_a^b f(x)\,dx = \int_a^a f(x)\,dx + \int_a^b f(x)\,dx$$

Thus,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 7.2.6 *Using the connection between integrals are area, evaluate* $\int_0^5 |x-2|\,dx$.

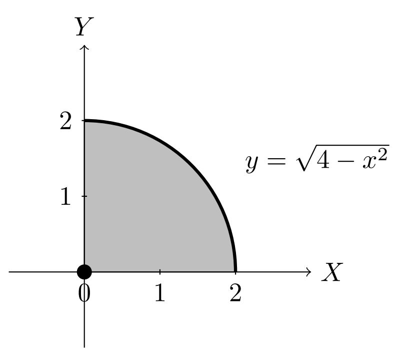
Solution. Define $f(x) = |x-2|$ where $x \in [0,5]$.



$$\int_0^5 f(x)\,dx = \int_0^5 |x-2|\,dx = \frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 3\cdot 3 = \frac{13}{2}$$

Example 7.2.7 *Using the connection between integrals are area, evaluate* $\int_0^2 \sqrt{4-x^2}\,dx$.

Solution. Define $f(x) = \sqrt{4-x^2}$ where $x \in [0,2]$.



$$\int_0^2 f(x)\,dx = \int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi(2)^2 = \pi$$

Theorem 7.2.8 (Comparison Theorem) *If f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In particular, if $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Proof. Assume that f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$.
Let P be a partition of $[a, b]$. By hypothesis, $M_j(f) \geq M_j(g)$ whence $U(f, P) \leq U(g, P)$.
It follows that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \leq U(f, P) \leq U(g, P)$$

for all partition P of $[a, b]$. Taking the infimum of this inequality over all partition P of $[a, b]$, we have

$$\int_a^b f(x)\,dx \leq \inf_P U(g, P) = (U)\int_a^b g(x)\,dx = \int_a^b g(x)\,dx.$$

If $m \leq f(x) \leq M$, then by Theorem 7.1.22

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a).$$

□

Theorem 7.2.9 *If f is Riemann integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. Assume that f is Riemann integrable on $[a, b]$. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$ and let $x, y \in [x_{j-1}, x_j]$ for $j = 1, 2, ..., n$. If $f(x), f(y)$ have the same sign, say both are positive, then

$$|f(x)| - |f(y)| = f(x) - f(y) \leq M_j(f) - m_j(f).$$

If $f(x), f(y)$ have opposite signs, $f(x) \geq 0 \geq f(y)$, then $m_j(f) \leq 0$, hence

$$|f(x)| - |f(y)| = f(x) + f(y) \leq M_j(f) + 0 \leq M_j(f) - m_j(f).$$

It implies that

$$M_j(|f|) - m_j(|f|) \leq M_j(f) - m_j(f). \tag{7.3}$$

Let $\varepsilon > 0$ and choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since (7.3) implies that $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$, it follows that

$$U(|f|, P) - L(|f|, P) < \varepsilon.$$

Thus, $|f|$ is Riemann integrable on $[a, b]$. Since $-|f(x)| \leq f(x) \leq |f(x)|$ holds for any $x \in [a, b]$, we conclude by Theorem 7.2.8 that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Hence, $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. □

Exercises 7.2

1. Using the connection between integrals are area, evaluate each of the following integrals.

   1.1 $\displaystyle\int_0^1 |x-0.5|\,dx$

   1.2 $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx, \quad a>0$

   1.3 $\displaystyle\int_{-2}^2 (|x+1|+|x|)\,dx$

   1.4 $\displaystyle\int_a^b (3x+1)\,dx, \quad a<b$

2. Prove that if f is integrable on $[0,1]$ and $\beta>0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{\frac{1}{n^\beta}} f(x)\,dx = 0 \quad \text{for all } \alpha<\beta.$$

3. If f,g are integrable on $[a,b]$ and $\alpha\in\mathbb{R}$, prove that

$$\left|\int_a^b (f(x)+g(x))\,dx\right| \le \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx.$$

4. Suppose that $g_n\ge 0$ is a sequence of integrable function that satisfies $\displaystyle\lim_{n\to\infty}\int_a^b g_n(x)\,dx=0$. Show that if $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$, then $\displaystyle\lim_{n\to\infty}\int_a^b f(x)g_n(x)\,dx=0$.

5. Prove that if f is integrable on $[0,1]$, then $\displaystyle\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx=0$.

6. Prove that if f is integrable on $[0,1]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=0}^n \int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} f(x)\,dx.$$

7. Let f be continuous on a closed, nondegenerate interval $[a,b]$ and set $M=\displaystyle\sup_{x\in[a,b]}|f(x)|$.

   7.1 Prove that if $M>0$ and $p>0$, then for every $\varepsilon>0$ there is a nondegenerate on interval $I\subset[a,b]$ such that

$$(M-\varepsilon)^p|I| \le \int_a^b |f(x)|^p\,dx \le M^p(b-a).$$

   7.2 Prove that $\displaystyle\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} = M.$

7.3 Fundamental Theorem of Calculus

Define a set $C^1[a,b] = \{f : [a,b] \to \mathbb{R} : f \text{ is differentiable and } f' \text{ are continuous } \}$ and $f'(x) = \dfrac{df}{dx}$.

Theorem 7.3.1 (**Fundamental Theorem of Calculus**) *Suppose that* $f : [a,b] \to \mathbb{R}$.

1. *If* f *is continuous on* $[a,b]$ *and* $F(x) = \int_a^x f(t)\,dt$, *then* $F \in C^1[a,b]$ *and*

$$\frac{d}{dx}\int_a^x f(t)\,dt := F'(x) = f(x)$$

   for each $x \in [a,b]$.

2. *If* f *is differentiable on* $[a,b]$ *and* f' *is integrable on* $[a,b]$, *then*

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

   for each $x \in [a,b]$.

Proof. Assume that f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ where $x \in [a,b]$.
Let $x_0 \in [a,b)$. Then $f(x) \to f(x_0)$ as $x \to x_0^+$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that

$$x_0 < t < x_0 + \delta \text{ and } t \in [a,b] \quad \text{imply} \quad |f(t) - f(x_0)| < \varepsilon. \tag{7.4}$$

Fix h such that $0 < h < \delta$. Use Theorem 7.1.22, We have

$$\begin{aligned}
\frac{F(x_0+h) - F(x_0)}{h} - f(x_0) &= \frac{1}{h}F(x_0+h) - \frac{1}{h}F(x_0) - \frac{1}{h}f(x_0)\cdot h \\
&= \frac{1}{h}\int_a^{x_0+h} f(t)\,dt - \frac{1}{h}\int_a^{x_0} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt \\
&= \frac{1}{h}\int_a^{x_0} f(t)\,dt + \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_a^{x_0} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt \\
&= \frac{1}{h}\int_{x_0}^{x_0+h} (f(t) - f(x_0))\,dt
\end{aligned}$$

By (7.4) and Theorem 7.2.9 , it implies that

$$\left|\frac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| \le \frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dx < \frac{1}{h}\int_{x_0}^{x_0+h} \varepsilon\,dx = \varepsilon.$$

Thus, $F'(x_0) = \lim\limits_{x \to x_0} \dfrac{F(x_0+h) - F(x_0)}{h} = f(x_0)$. The proof of part 1 is complete.

2. Assume that f is differentiable on $[a,b]$ and f' is integrable on $[a,b]$. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that

$$\left|\sum_{j=1}^{n} f'(t_j)\Delta x_j - \int_a^b f'(x)\,dx\right| < \varepsilon$$

for any choice of points $t_j \in [x_{j-1}, x_j]$. Use the MVT to choose points $t_j \in [x_{j-1}, x_j]$ such that

$$f(x_j) - f(x_{j-1}) = f'(t_j)(x_j - x_{j-1}) = f'(t_j)\Delta x_j.$$

It follows by telescoping that $\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) = f(b) - f(a)$ and

$$\begin{aligned} \left|f(b) - f(a) - \int_a^b f'(t)\,dt\right| &= \left|\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) - \int_a^b f'(t)\,dt\right| \\ &= \left|\sum_{j=1}^{n} f'(t_j)\Delta x_j - \int_a^b f'(t)\,dt\right| < \varepsilon. \end{aligned}$$

Thus, $\int_a^b f'(t)\,dt = f(b) - f(a)$ for case $x = b$. It suffices to prove part 2. □

Example 7.3.2 *Assume that f is differentiable on $(0,1)$ and integrable on $[0,1]$. Show that*

$$\int_0^1 xf'(x) + f(x)\,dx = f(1).$$

Solution. By the Product Rule, we have $(xf(x))' = xf'(x) + f(x)$.

Apply the Fundamental Theorem of Calculus,

$$\int_0^1 xf'(x) + f(x)\,dx = \int_0^1 (xf(x))'\,dx = 1f(1) - 0f(0) = f(1).$$

Theorem 7.3.3 *Let $\alpha \neq -1$. Then*

$$\int_a^b x^\alpha\,dx = f(b) - f(a) \quad \textit{where } f(x) = \frac{x^{\alpha+1}}{\alpha+1}.$$

Proof. Let $\alpha \neq -1$. The $f'(x) = x^\alpha$. By part 2 of the Fundamental Theorem of Calculus, we obtain this Theorem. □

Example 7.3.4 *Find integral $\int_0^1 x^2\,dx$.*

Solution. By the Power Rule, we have

$$\int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Theorem 7.3.5 *Suppose that* $f, u : [a, b] \to \mathbb{R}$. *If* f *is continuous on* $[a, b]$ *and* $F(x) = \int_a^{u(x)} f(t)\,dt$, *and* $F \in C^1[a, b]$ *and*

$$F'(x) = \frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x)) \cdot u'(x)$$

for each $x \in [a, b]$.

Proof. Apply the Chain Rule. ☐

Example 7.3.6 *Let* $F(x) = \int_0^{\sin x} e^{t^2}\,dt$. *Find* $F(0)$ *and* $F'(0)$.

Solution. We obtain $F(0) = \int_0^0 e^{t^2}\,dt = 0$ and by Theorem 7.3.5, it implies that

$$F'(x) = \frac{d}{dx}\int_0^{\sin x} e^{t^2}\,dt = e^{(\sin x)^2} \cdot (\sin x)' = e^{\sin^2 x} \cdot \cos x.$$

Thus, $F'(0) = 1$.

INTEGRATION BY PART.

Theorem 7.3.7 (**Integration by Part**) *Suppose that* f, g *are differentiable on* $[a, b]$ *with* f', g' *integrable on* $[a, b]$, *Then*

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Proof. Assume that f, g are differentiable on $[a, b]$ with f', g' integrable on $[a, b]$. By the Product Rule, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ for $x \in [a, b]$. It implies that $(fg)'$ is integrable on $[a, b]$. Thus, by the part 2 of the Fundamental Theorem of Calculus, we obtain

$$\int_a^b f'(x)g(x)\,dx = \int_a^b (fg)'(x)\,dx - \int_a^b f(x)g'(x)\,dx$$
$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

The proof is complete. ☐

Example 7.3.8 *Use the Integration by Part to find integrals.*

1. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$
2. $\displaystyle\int_1^2 \ln x\,dx$

Solution. By the Integration by Part and The Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x\sin x\,dx &= \int_0^{\frac{\pi}{2}} x(-\cos x)'\,dx = \frac{\pi}{2}(-\cos\frac{\pi}{2}) - 0(-\cos 0) - \int_0^{\frac{\pi}{2}} (x)'(-\cos x)\,dx\\
&= \int_0^{\frac{\pi}{2}} \cos x\,dx = \int_0^{\frac{\pi}{2}} (\sin x)'\,dx = \sin\frac{\pi}{2} - \sin 0 = 1.\\
\int_1^2 \ln x\,dx &= \int_1^2 (x)'\ln x\,dx = 2\ln 2 - 1\ln 1 - \int_1^2 x(\ln x)'\,dx\\
&= 2\ln 2 - \int_1^2 x\cdot\frac{1}{x}\,dx = 2\ln 2 - \int_1^2 1\,dx\\
&= 2\ln 2 - \int_1^2 (x)'\,dx = 2\ln 2 - (2-1) = 2\ln 2 - 1.
\end{aligned}$$

Example 7.3.9 *Let* $f(x) = \displaystyle\int_0^{x^3} e^{t^2}\,dt$. *Use integration by part to show that*

$$6\int_0^1 x^2 f(x)dx - 2\int_0^1 e^{x^2}dx = 1-e.$$

Solution. By the Theorem 7.3.5, $f'(x) = e^{(x^3)^2}\cdot(x^3)' = 3x^2e^{x^6}$. We obtain

$$\begin{aligned}
6\int_0^1 x^2 f(x)dx &= 2\int_0^1 (3x^2)f(x)dx\\
&= 2\int_0^1 (x^3)'f(x)dx\\
&= 2\left(1f(1) - 0f(0) - \int_0^1 x^3 f'(x)dx\right)\\
&= 2\left(f(1) - \int_0^1 x^3(3x^2e^{x^6})dx\right)\\
&= 2f(1) - \int_0^1 6x^5e^{x^6}dx\\
&= 2\int_0^1 e^{x^2}dx - \int_0^1 (e^{x^6})'dx\\
&= 2\int_0^1 e^{x^2}dx - [e-1]
\end{aligned}$$

We conclude that $\displaystyle 6\int_0^1 x^2 f(x)dx - 2\int_0^1 e^{x^2}dx = 1-e.$

CHANGE OF VARIABLES.

Theorem 7.3.10 (**Change of Variables**) *Let ϕ be continuously differentiable on a closed interval $[a, b]$. If f is continuous on $\phi([a, b])$, or if ϕ is strictly incresing on $[a, b]$ and f is integrable on $[\phi(a), \phi(b)]$, then*

$$\int_{\phi(a)}^{\phi(b)} f(t)\, dt = \int_a^b f(\phi(x))\phi'(x)\, dx.$$

Proof. Exercise. □

Example 7.3.11 *Find* $\int_0^3 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx$

Solution. Let $f(x) = e^x$ and $\phi(x) = \sqrt{x+1}$ where $x \in [0, 3]$. Then $\phi'(x) = \frac{1}{2\sqrt{x+1}}$ such that $\phi(0) = 1$ and $\phi(3) = 2$. It follows that

$$f(\phi(x)) \cdot \phi'(x) = \frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}.$$

By the Change of Variables, we obtain

$$\begin{aligned}\int_0^3 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx &= 2\int_0^3 f(\phi(x)) \cdot \phi'(x)\, dx = 2\int_{\phi(0)}^{\phi(3)} f(t)\, dt \\ &= 2\int_1^2 e^t\, dt = 2\int_1^2 (e^t)'\, dt = 2(e^2 - e).\end{aligned}$$

Example 7.3.12 *Evaluate*

$$\int_{-1}^{1} x f(x^2)\, dx$$

for any f is continuous on $[0, 1]$.

Solution. Let $\phi(x) = x^2$ where $x \in [-1, 1]$. Then $\phi'(x) = 2x$ such that $\phi(-1) = 1$ and $\phi(1) = 1$. It follows that

$$f(\phi(x)) \cdot \phi'(x) = f(x^2) \cdot 2x.$$

By the Change of Variables, we obtain

$$\int_{-1}^{1} x f(x^2)\, dx = \frac{1}{2}\int_{-1}^{1} f(\phi(x)) \cdot \phi'(x)\, dx = \frac{1}{2}\int_{\phi(-1)}^{\phi(1)} f(t)\, dt = \frac{1}{2}\int_1^1 f(t)\, dt = 0$$

Example 7.3.13 *Let $f : [-a, a] \to \mathbb{R}$ where $a > 0$. Suppose $f(-x) = -f(x)$ for all $x \in [-a, a]$. Show that*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Solution. Let $\phi(x) = -x$ where $x \in [-a, a]$. Then $\phi'(x) = -1$ such that $\phi(-a) = a$ and $\phi(a) = -a$. It follows by the Change of Variables that

$$\begin{aligned}
\int_{-a}^{a} f(x)\, dx &= \int_{-a}^{a} -f(x) \cdot (-1)\, dx \\
&= \int_{-a}^{a} f(-x) \cdot \phi'(x)\, dx \\
&= \int_{-a}^{a} f(\phi(x)) \cdot \phi'(x)\, dx \\
&= \int_{\phi(-a)}^{\phi(a)} f(t)\, dt \\
&= \int_{a}^{-a} f(t)\, dt \\
&= -\int_{-a}^{a} f(t)\, dt.
\end{aligned}$$

Then, $2\int_{-a}^{a} f(x)\, dx = 0$. We conclude that $\int_{-a}^{a} f(x)\, dx = 0$.

Exercises 7.3

1. Compute each of the following integrals.

   1.1 $\displaystyle\int_{-3}^{3} |x^2+x-2|\,dx$

   1.2 $\displaystyle\int_{1}^{4} \frac{\sqrt{x}-1}{\sqrt{x}}\,dx$

   1.3 $\displaystyle\int_{0}^{1} (3x+1)^{99}\,dx$

   1.4 $\displaystyle\int_{1}^{e} x\ln x\,dx$

   1.5 $\displaystyle\int_{0}^{\frac{\pi}{2}} e^x\sin x\,dx$

   1.6 $\displaystyle\int_{0}^{1} \sqrt{\frac{4x^2-4x+1}{x^2-x+3}}\,dx$

2. Use First Mean Value Theorem for Integrals to prove the followingversion of the Mean Value Theorem for Derivatives. If $f \in C^1[a,b]$, then there is an $x_0 \in [a,b]$ such that
$$f(b)-f(a) = (b-a)f'(x_0).$$

3. If $f : [0,\infty) \to \mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dx}\int_0^{x^2} f(t)\,dt$.

4. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dt}\int_{\cos t}^{t} g(x)\,dx$.

5. Let g be differentiable and integrable on $\mathbb{R}$. Define $f(x) = \displaystyle\int_1^{x^2} g(t)\cdot\sqrt{t}\,dt$.

   Show that $\displaystyle\int_0^1 xg(x)+f(x)\,dx = 0$.

6. If $f(x) = \displaystyle\int_0^{x^2} \sec^2(t^2)dt$. show that $\displaystyle 2\int_0^1 \sec^2(x^2)\,dx - 4\int_0^1 xf(x)\,dx = \tan 1$.

7. Suppose that g is integrable and nonnegative on $[1,3]$ with $\displaystyle\int_1^3 g(x)\,dt = 1$. Prove that
$$\frac{1}{\pi}\int_1^9 g(\sqrt{x})\,dx < 2.$$

8. Suppose that h is integrable and nonnegative on $[1,11]$ with $\displaystyle\int_1^{11} h(x)\,dt = 3$. Prove that
$$\int_0^2 h(1+3x+3x^2-x^3)\,dx \le 1.$$

9. If f is continuous on $[a,b]$ and there exist numbers $\alpha \neq \beta$ such that
$$\alpha\int_a^c f(x)\,dx + \beta\int_c^b f(x)\,dx = 0$$
holds for all $c \in (a,b)$, prove that $f(x) = 0$ for all $x \in [a,b]$.

Chapter 8

Infinite Series of Real Numbers

8.1 Introduction

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of numbers. We shall call an expression of the form

$$\sum_{k=1}^{\infty} a_k$$

an **infinite series** with terms a_k.

Definition 8.1.1 *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series whose terms* a_k *belong to* $\mathbb{R}$.

1. *The* ***partial sums*** *of* S *of order* n *are the numbers defined, for each* $n \in \mathbb{N}$, *by*

$$s_n := \sum_{k=1}^{n} a_k.$$

2. S *is said to* ***converge*** *if and only if its sequence of partial sums* $\{s_n\}$ *to some* $s \in \mathbb{R}$ *as* $n \to \infty$*; i.e., for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$n \geq N \quad \textit{implies} \quad |s_n - s| < \varepsilon.$$

In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call s *the sum, or value, of the series* $\sum_{k=1}^{\infty} a_k$.

3. S *is said to* ***diverge*** *if and only if its sequence of partial sums* $\{s_n\}$ *does not converge.*

Example 8.1.2 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]=1.$

Solution. Use telescoping, we have

$$s_n=\sum_{k=1}^{n}\left[\frac{1}{k}-\frac{1}{k+1}\right]=1-\frac{1}{n+1}.$$

Then, $\displaystyle\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1.$ We conclude that $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]=1.$

Example 8.1.3 *Prove that* $\displaystyle\sum_{k=1}^{\infty}(-1)^k$ *diverges.*

Solution. We see that

$$s_n=\sum_{k=1}^{n}(-1)^k=\begin{cases}-1 & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

It is easy to see that s_n does not converge as $n\to\infty$. Hence, $\displaystyle\sum_{k=1}^{\infty}(-1)^k$ diverges.

Theorem 8.1.4 (**Harmonic Series**) *Prove that the sequence* $\frac{1}{k}$ *converges but the series*

$$\sum_{k=1}^{\infty}\frac{1}{k}\qquad \textit{diverges.}$$

Proof. By Example 2.1.5, it implies that $\frac{1}{k}\to 0$ as $k\to\infty$. Let $x\in[k,k+1]$ for each $k\in\mathbb{N}$. Then

$$\frac{1}{k+1}\le\frac{1}{x}\le\frac{1}{k}.$$

By Comparison Theorem for integral, We obtain

$$\int_k^{k+1}\frac{1}{x}\,dx\le\int_k^{k+1}\frac{1}{k}\,dx=\frac{1}{k}$$

It follows that

$$s_n=\sum_{k=1}^{n}\frac{1}{k}\ge\sum_{k=1}^{n}\int_k^{k+1}\frac{1}{x}\,dx=\int_1^{n+1}\frac{1}{x}\,dx=\ln(n+1)$$

We conclude that $s_n\to\infty$ as $n\to\infty$, i.e., $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. □

Theorem 8.1.5 (**Divergence Test**) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *be a sequence of real numbers.*

$$\textit{If } a_k \textit{ does not converge to zero, then the series } \sum_{k=1}^{\infty} a_k \textit{ diverges.}$$

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges and equals to s. Then

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and } s_n \to s \text{ as } n \to \infty.$$

Since $a_k = s_{k+1} - s_k$,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (s_{k+1} - s_k) = s - s = 0.$$

Thus, a_k converges to zero. □

Example 8.1.6 *Show that the series* $\sum_{k=1}^{\infty} \frac{n}{n+1}$ *diverges.*

Solution. We see that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

By the Divergence Test, it imlplies that $\sum_{k=1}^{\infty} \frac{n}{n+1}$ diverges.

Theorem 8.1.7 (**Telescopic Seires**) *If* $\{a_k\}$ *is a convergent real sequence, then*

$$\sum_{k=m}^{\infty} (a_k - a_{k+1}) = a_m - \lim_{k\to\infty} a_k.$$

Proof. By telescoping, we have

$$s_n = \sum_{k=m}^{n} (a_k - a_{k+1}) = a_m - a_{n+1}.$$

Thus,

$$\begin{aligned}\sum_{k=m}^{\infty} (a_k - a_{k+1}) &= \lim_{n\to\infty} (a_m - a_{n+1}) \\ &= a_m - \lim_{n\to\infty} a_{n+1} \\ &= a_m - \lim_{k\to\infty} a_k.\end{aligned}$$

□

Example 8.1.8 *Evaluate the series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$.

Solution. By the Telescopic Series, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$
$$= \frac{1}{1+1} - \lim_{k\to\infty} \frac{1}{k+1} = \frac{1}{2}.$$

Example 8.1.9 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}}$ *converges or not.*

Solution. Use telescoping, we have

$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n} \left[\sqrt{k+1} - \sqrt{k} \right] = \sqrt{n+1} - 1.$$

Then, $s_n \to \infty$ as $n \to \infty$. We conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}}$ diverges.

Theorem 8.1.10 (**Geometric Seires**) *The series* $\displaystyle\sum_{k=1}^{\infty} x^k$ *converges if and only if* $|x| < 1$, *in which case*

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}.$$

Proof. If $|x| \geq 1$, then $\{x^k\}$ diverges. By The Divergence Test, it implies that $\displaystyle\sum_{k=1}^{\infty} x^k$ diverges.
Case $|x| < 1$. Then $x^k \to 0$ as $k \to \infty$. Since $x^k - x^{k+1} = x^k(1-x)$, we have

$$x^k = \frac{x^k}{1-x} - \frac{x^{k+1}}{1-x}.$$

By the Telescopic Series,

$$\sum_{k=1}^{\infty} x^k = \sum_{k=1}^{\infty} \left[\frac{x^k}{1-x} - \frac{x^{k+1}}{1-x} \right]$$
$$= \frac{x}{1-x} - \lim_{k\to\infty} \frac{x^k}{1-x}$$
$$= \frac{x}{1-x}$$

Thus, $\displaystyle\sum_{k=1}^{\infty} x^k$ converges if and only if $|x| < 1$ and $\displaystyle\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. □

Example 8.1.11 *Determine whether the following series converges or diverges.*

1. $\sum_{k=1}^{\infty} 2^{-k}$
2. $\sum_{k=1}^{\infty} (\sqrt{2}-1)^{-k}$

Solution. For 1. We have $x = \frac{1}{2}$ such that $|x| < 1$. It implies that $\sum_{k=1}^{\infty} 2^{-k}$ converges and

$$\sum_{k=1}^{\infty} 2^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

Since $\sum_{k=1}^{\infty} (\sqrt{2}-1)^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}-1}\right)^k$ and $\frac{1}{\sqrt{2}-1} = \sqrt{2}+1 > 1$,

we conclude that $\sum_{k=1}^{\infty} (\sqrt{2}-1)^{-k}$ diverges.

Theorem 8.1.12 *Let $\{a_k\}$ and $\{b_k\}$ be a real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then*

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \quad \textit{and} \quad \sum_{k=1}^{\infty} (\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$

for any $\alpha \in \mathbb{R}$.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. Assume that $s_n \to s$ and $t_n \to t$ as $n \to \infty$. Then

$$s_n + t_n = \sum_{k=1}^{n} (a_k + b_k) \quad \text{and} \quad \alpha s_n = \sum_{k=1}^{n} \alpha a_k.$$

By the Limit Theorem, it implies that $s_n + t_n \to s + t$ and $\alpha s_n \to \alpha s$ as $n \to \infty$.
The proof of this Theorem is complete. □

Theorem 8.1.13 *If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then*

$$\sum_{k=1}^{\infty} (a_k + b_k) \ \textit{diverges.}$$

Proof. Exercise. □

Example 8.1.14 *Evaluate* $\displaystyle\sum_{k=1}^{\infty}\frac{1+2^{k+1}}{3^k}$.

Solution. Use the Geometric Series and Theorem 8.1.12, it implies that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1+2^{k+1}}{3^k} &= 2+\sum_{k=1}^{\infty}\frac{1+2^{k+1}}{3^k} = 2+\sum_{k=1}^{\infty}\left[\left(\frac{1}{3}\right)^k+2\left(\frac{2}{3}\right)^k\right]\\ &= 2+\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k+2\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k\\ &= 2+\frac{\frac{1}{3}}{1-\frac{1}{3}}+2\cdot\frac{\frac{2}{3}}{1-\frac{2}{3}} = 2+\frac{1}{2}+4=\frac{13}{2}.\end{aligned}$$

Example 8.1.15 *Evaluate* $\displaystyle\sum_{k=1}^{\infty}\frac{k}{2^k}$.

Solution. Consider the difference of

$$\frac{k}{2^k}-\frac{1}{2^k}=\frac{k-1}{2^k}=\frac{2k-k-1}{2^k}=\frac{2k}{2^k}-\frac{k+1}{2^k}=\frac{k}{2^{k-1}}-\frac{k+1}{2^k}.$$

By the Telescopic and Geometric Series, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{k}{2^k} &= \sum_{k=1}^{\infty}\left[\frac{1}{2^k}+\left(\frac{k}{2^{k-1}}-\frac{k+1}{2^k}\right)\right]\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}+\sum_{k=1}^{\infty}\left[\frac{k}{2^{k-1}}-\frac{k+1}{2^k}\right]\\ &= \frac{\frac{1}{2}}{1-\frac{1}{2}}+1-\lim_{k\to\infty}\frac{k+1}{2^k}=1+1-0=2\end{aligned}$$

Example 8.1.16 *Let π be a Pi constant. Show that*

$$\sum_{k=1}^{\infty}\frac{1}{\pi^{k^2}}\left[1-\frac{\pi^{2k}}{\pi}+\left(\frac{\pi^k}{\pi}\right)^k\right]$$

converges and find its value.

Solution. We rewrite the term of this series

$$\begin{aligned}\frac{1}{\pi^{k^2}}\left[1-\frac{\pi^{2k}}{\pi}+\left(\frac{\pi^k}{\pi}\right)^k\right] &= \frac{1}{\pi^{k^2}}-\frac{1}{\pi^{k^2-2k+1}}+\frac{1}{\pi^{k^2}}\cdot\frac{\pi^{k^2}}{\pi^k}\\ &= \left(\frac{1}{\pi^{k^2}}-\frac{1}{\pi^{(k-1)^2}}\right)+\left(\frac{1}{\pi}\right)^k\end{aligned}$$

Then, the first term is telescoping series and the second term is geometric series. Thus,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi}\right)^k\right] &= \sum_{k=1}^{\infty} \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}}\right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi}\right)^k \\
&= -\sum_{k=1}^{\infty} \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{k^2}}\right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi}\right)^k \\
&= -1 + \lim_{k\to\infty} \frac{1}{\pi^{k^2}} + \frac{\frac{1}{\pi}}{1 - \frac{1}{\pi}} \\
&= -1 + 0 + \frac{1}{\pi - 1} = \frac{2-\pi}{\pi - 1} \quad \#
\end{aligned}$$

Example 8.1.17 *Evaluate the series* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$.

Solution. By the Telescopic Series, we obtain

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{k^2-1} &= \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} \\
&= \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\
&= \frac{1}{2} \sum_{k=2}^{\infty} \left[\left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right)\right] \\
&= \frac{1}{2} \left[\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) + \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)\right] \\
&= \frac{1}{2} \left[\left(1 - \lim_{k\to\infty} \frac{1}{k}\right) + \left(\frac{1}{2} - \lim_{k\to\infty} \frac{1}{k+1}\right)\right] \\
&= \frac{1}{2} \left[1 - 0 + \frac{1}{2} - 0\right] = \frac{3}{4}.
\end{aligned}$$

Example 8.1.18 *Evaluate* $\displaystyle\sum_{k=2}^{\infty} \left(\frac{1}{n^2-1} + \frac{2^k}{7\cdot 5^k}\right)$.

Solution. Use Example 8.1.17, it implies that

$$\begin{aligned}
\sum_{k=2}^{\infty} \left(\frac{1}{n^2-1} + \frac{2^k}{7\cdot 5^k}\right) &= \sum_{k=2}^{\infty} \frac{1}{n^2-1} + \sum_{k=2}^{\infty} \frac{2^k}{7\cdot 5^k} \\
&= \frac{3}{4} + \frac{1}{7} \sum_{k=2}^{\infty} \left(\frac{2}{5}\right)^k \\
&= \frac{3}{4} + \frac{1}{7} \cdot \frac{\frac{4}{25}}{1 - \frac{2}{5}} = \frac{3}{4} + \frac{4}{105} = \frac{331}{420}
\end{aligned}$$

Exercises 8.1

1. Show that
$$\sum_{k=n}^{\infty} x^k = \frac{x^n}{1-x}$$
for $|x| < 1$ and $n = 0, 1, 2, \ldots$.

2. Prove that each of the following series converges and find its value.

   2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}}$

   2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi^k}$

   2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}+4}{5^k}$

   2.4 $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{7^{k-1}}$

   2.5 $\displaystyle\sum_{k=0}^{\infty} 2^k e^{-k}$

   2.6 $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{2^k}$

3. Represent each of the following series as a telescopic series and find its value.

   3.1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$

   3.2 $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right)$

   3.3 $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{\frac{\pi}{4}}\left(1-\left(\frac{\pi}{4}\right)^{j_k}\right)$, where $j_k = -\dfrac{1}{k(k+1)}$ for $k \in \mathbb{N}$

4. Find all $x \in \mathbb{R}$ for which
$$\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1})$$
converges. For each such x, find the value of this series.

5. Prove that each of the following series diverges.

   5.1 $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k^2}$

   5.2 $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^k$

   5.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

6. Prove that if $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

7. Let $\{b_k\}$ be a real sequence and $b \in \mathbb{R}$.

7.1 Suppose that there is an $N \in \mathbb{N}$ such that $|b - b_k| \le M$ for all $k \ge N$. Prove that

$$\left| nb - \sum_{k=1}^{n} b_k \right| \le \sum_{k=1}^{N} |b_k - b| + M(n - N)$$

for all $n > N$.

7.2 Prove that if $b_k \to b$ as $k \to \infty$, then

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \to b \quad \text{as} \quad n \to \infty.$$

7.3 Show that converse of 7.2 is false.

8. A series $\sum_{k=0}^{\infty} a_k$ is said to be **Cesàro summable** to $L \in \mathbb{R}$ if and only if

$$\sigma_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k$$

converges to L as $n \to \infty$.

8.1 Let $s_n = \sum_{k=0}^{\infty} a_k$. Prove that $\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$ for each $n \in \mathbb{N}$.

8.2 Prove that if $a_k \in \mathbb{R}$ and $\sum_{k=0}^{\infty} a_k = L$ converges, then c is Cesàro summable to L.

8.3 Prove that $\sum_{k=0}^{\infty} (-1)^k$ is Cesàro summable to $\frac{1}{2}$; hence the converge of 8.2 is false.

8.4 **TAUBER.** Prove that if $a_k \ge 0$ for $k \in \mathbb{N}$ and $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L, then $\sum_{k=0}^{\infty} a_k = L$.

9. Suppose that $\{a_k\}$ is a decreasing sequence of real numbers. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $ka_k \to 0$ as $k \to \infty$.

10. Suppose that $a_k \ge 0$ for k large and $\sum_{k=0}^{\infty} \frac{a_k}{k}$ converges. Prove that $\lim_{j \to \infty} \sum_{k=1}^{\infty} \frac{a_k}{j + k} = 0$.

11. If and $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges.

8.2 Series with nonnegative terms

INTEGRAL TEST.

Theorem 8.2.1 (**Integral Test**) *Suppose that* $f : [1, \infty) \to \mathbb{R}$ *is positive and decreasing on* $[1, \infty)$. *Then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if*

$$\lim_{n\to\infty} \int_1^n f(x)\,dx < \infty.$$

Proof. Let $s_n = \sum_{k=1}^{n} f(k)$ and $t_n = \int_1^n f(x)\,dx$ for $n \in \mathbb{N}$. Since f is positive and decreasing on $[1, \infty)$, f is locally integrable on $[1, \infty)$. For each $k \in \mathbb{N}$, we have

$$f(k+1) \leq f(x) \leq f(k) \quad \text{for all } x \in [k, k+1].$$

Taking integrate on $[k, k+1]$, we obtain

$$f(k+1) = \int_k^{k+1} f(k+1)\,dx \leq \int_k^{k+1} f(x)\,dx \leq \int_k^{k+1} f(k)\,dx = f(k).$$

Summing over $k = 1, 2, ..., n-1$, it follows that

$$\begin{array}{ccccc}
\sum_{k=1}^{n-1} f(k+1) & \leq & \sum_{k=1}^{n-1} \int_k^{k+1} f(k+1)\,dx & \leq & \sum_{k=1}^{n-1} f(k) \\
s_n - f(1) & \leq & \int_1^n f(k+1)\,dx & \leq & s_n - f(n) \\
s_n - f(1) & \leq & t_n & \leq & s_n - f(n) \\
-f(1) & \leq & t_n - s_n & \leq & -f(n) \\
f(n) & \leq & s_n - t_n & \leq & f(1)
\end{array}$$

Thus, $\{s_n\}$ is bounded if and only if $\{t_n\}$ is. Since f is positir, it implies that both s_n and t_n are incresing. It follows that from the Monotone Convergence Theorem that s_n converges if and only if t_n converges. $\square$

Example 8.2.2 *Use the Integral Test to prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

Solution. Let $f(x) = \dfrac{1}{x}$. Then f is positive and decreasing on $[1, \infty)$. We obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_1^n f(x)\, dx &= \lim_{n\to\infty} \int_1^n \frac{1}{x}\, dx \\
&= \lim_{n\to\infty} \int_1^n (\ln x)'\, dx \\
&= \lim_{n\to\infty} (\ln n - \ln 1) = \infty.
\end{aligned}$$

By the Integral Test, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Example 8.2.3 *Show that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges.*

Solution. Let $f(x) = \dfrac{1}{x^2}$. Then f is positive and decreasing on $[1, \infty)$. We obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_1^n f(x)\, dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\, dx &= \lim_{n\to\infty} \int_1^n (-x^{-1})'\, dx \\
&= \lim_{n\to\infty} \left(-\frac{1}{n} + 1\right) = 1 < \infty.
\end{aligned}$$

By the Integral Test, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Example 8.2.4 *Show that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *converges.*

Solution. Let $f(x) = \dfrac{1}{x^2+1}$. Then f is positive and decreasing on $[1, \infty)$. We obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_1^n f(x)\, dx &= \lim_{n\to\infty} \int_1^n \frac{1}{x^2+1}\, dx \\
&= \lim_{n\to\infty} \int_1^n (\arctan x)'\, dx \\
&= \lim_{n\to\infty} (\arctan n - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty.
\end{aligned}$$

By the Integral Test, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges.

---

p-SERIES TEST.

Theorem 8.2.5 (p-Series Test) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Proof. If $p < 0$ or $p = 1$, then the series diverges. Case $p > 0$ and $p \neq 1$, set $f(x) = x^{-p}$ and observe that

$$f'(x) = -px^{-p-1} < 0 \text{ for all } x \in [1, \infty).$$

Thus, f is positive and decreasing on $[1, \infty)$. Since

$$\lim_{n\to\infty} \int_1^n x^{-p}\, dx = \lim_{n\to\infty} \int_1^n \left(\frac{x^{1-p}}{1-p}\right)' dx = \lim_{n\to\infty} \frac{n^{1-p}-1}{1-p}$$

has a finite limit if and only if $1 - p < 0$. It follows from the Integral Test that p-series converges if and only if $p > 1$. $\square$

Example 8.2.6 *Find* $p \in \mathbb{R}$ *such that* $\displaystyle\sum_{k=1}^{\infty} k^{p^2-2}$ *converges.*

Solution. Rewrite the sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{2-p^2}}$ which is a p-series. Then the series converges if and only if $2 - p^2 > 1$. It follows that $p^2 - 1 < 0$ is equivalent to $p \in (-1, 1)$

Example 8.2.7 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2^k}{k2^k}\right)$ *converges or not.*

Solution. Consider

$$\frac{k+2^k}{k2^k} = \frac{1}{2^k} + \frac{1}{k}.$$

Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (the p-Series Test, $p = 1$) and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$ converge (the geometric series, $x = \frac{1}{2}$), we conclude that

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{2^k}\right) = \sum_{k=1}^{\infty} \left(\frac{k+2^k}{k2^k}\right) \quad \text{diverges.}$$

COMPARISON TEST.

Theorem 8.2.8 *Suppose that* $a_k \geq 0$ *for* $k \geq N$. *Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if its sequence of partial sums* $\{s_n\}$ *is bounded, i.e., if and only if there exists a finite number* $M > 0$ *such that*

$$\left|\sum_{k=1}^{n} a_k\right| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ for $n \in \mathbb{N}$. If $\sum_{k=1}^{\infty} a_k$ converges, then s_n convergess as $n \to \infty$. Since every convergent sequence is bounded by the BCT, s_n is bounded. The proof is complete. □

Theorem 8.2.9 (**Comparison Test**) *Suppose that there is an* $M \in \mathbb{N}$ *such that*

$$0 \leq a_k \leq b_k \quad \text{for all } k \geq M.$$

1. *If* $\sum_{k=1}^{\infty} b_k < \infty$, *then* $\sum_{k=1}^{\infty} a_k < \infty$.
2. *If* $\sum_{k=1}^{\infty} a_k = \infty$, *then* $\sum_{k=1}^{\infty} b_k = \infty$.

Proof. Assume that there is an $M \in \mathbb{N}$ such that $0 \leq a_k \leq b_k$ for all $k \geq M$.
Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. For each $n \geq M$, we sum over $k = M+1, ..., n$

$$0 \leq \sum_{k=M+1}^{n} a_k \leq \sum_{k=M+1}^{n} b_k$$

$$0 \leq s_n - s_M \leq t_n - t_M.$$

Since M is fixed, it follows that s_n is bounded when t_n is, t_n is unbounded when s_n is. Apply Theorem 8.2.8, we obtain this Theorem. □

Example 8.2.10 *Determine whether the following series converges or diverges.*

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+3^k}$

Solution. Since $k^3+1 > k^3 > 0$ and $3^k + k^3 > k^3 > 0$ for all $k \in \mathbb{N}$, we have

$$0 < \frac{1}{k^3+1} < \frac{1}{k^3} \quad \text{and} \quad 0 < \frac{1}{k^3+3^k} < \frac{1}{k^3}.$$

We see that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the p-Series Test ($p = 3 > 1$). It implies by the Comparison Test that

$$\sum_{k=1}^{\infty} \frac{1}{k^3+1} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^3+3^k} \text{ converge.}$$

Example 8.2.11 *Determine whether* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ *converges or diverges.*

Solution. Use the MVT to prove that (see 1.10 of Exercise 6.3)

$$\ln x \leq \sqrt{x} \quad \text{for all } x > 1.$$

It follows that $0 < \ln k \leq \sqrt{k}$ for all $k > 1$. Then

$$0 < \frac{1}{\sqrt{k}} < \frac{1}{\ln k}.$$

We see that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by the p-Series Test ($p = \frac{1}{2} < 1$). It implies by the Comparison Test that

$$\sum_{k=2}^{\infty} \frac{1}{\ln k} \quad \text{diverges.}$$

LIMIT COMPARISON TEST.

Theorem 8.2.12 (**Limit Comparison Test**) *Suppose that a_k and b_k are positive for lagre k and*

$$L := \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists as an extended real number.

1. *If $0 < L < \infty$, then $\sum_{k=1}^{\infty} b_k$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
3. *If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. Assume that a_k and b_k are positive for lagre k and $\dfrac{a_k}{b_k} \to L$ as $k \to \infty$.

1. Case $0 < L < \infty$. Given $\varepsilon = \dfrac{L}{2}$. There is an $N \in \mathbb{N}$ such that

$$k \geq N \quad \text{implies} \quad \left|\frac{a_k}{b_k} - L\right| < \frac{L}{2}.$$

For each $n \geq N$, we have $-\dfrac{L}{2} < \dfrac{a_k}{b_k} - L < \dfrac{L}{2}$, i.e.,

$$0 < \frac{L}{2} \cdot b_k < a_k < \frac{3L}{2} \cdot b_k.$$

Hence, part 1 follows immediately from the Comparison Test and Theorem 8.1.12.

Similar arguments establish part 2 and 3. □

Example 8.2.13 *Use the Limit Comparison Test to prove that $\sum_{k=1}^{\infty} \dfrac{1}{k^2+1}$ converge.*

Solution. Let $a_k = \dfrac{1}{x^2+1}$ and $b_k = \dfrac{1}{k^2}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 < \infty.$$

We see that $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ converges by the p-Series Test ($p = 2 > 1$). It implies by the Limit Comparison Test that

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1} \quad \text{converges.}$$

Example 8.2.14 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2k^4+k+3}$ *converges or diverges.*

Solution. Let $a_k = \dfrac{k}{2k^4+k+3}$ and $b_k = \dfrac{1}{k^3}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^4}{2k^4+k+3} = \frac{1}{2} < \infty.$$

We see that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the p-Series Test $(p = 3 > 1)$. It implies by the Limit Comparison Test that $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2k^4+k+3}$ converges.

Example 8.2.15 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ *converges or diverges.*

Solution. Let $a_k = \dfrac{1}{\sqrt{k}+1}$ and $b_k = \dfrac{1}{\sqrt{k}}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = 1 < \infty.$$

We see that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by the p-Series Test $(p = \frac{1}{2} < 1)$. It implies by the Limit Comparison Test that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ diverges.

Theorem 8.2.16 *Let $a_k \to 0$ as $k \to \infty$. Prove that*

$$\sum_{k=1}^{\infty} \sin|a_k| \text{ converges} \quad \text{if and only if} \quad \sum_{k=1}^{\infty} |a_k| \text{ converges.}$$

Proof. Assume that $a_k \to 0$ as $k \to \infty$. We will see that

$$\lim_{k\to\infty} \frac{\sin|a_k|}{|a_k|} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 < \infty.$$

By the Limit comparison Test, it implies that $\displaystyle\sum_{k=1}^{\infty} \sin|a_k|$ converges if and only if $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges.

□

Exercises 8.2

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty} \frac{k-3}{k^3+k+1}$

1.2 $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k2^k}$

1.3 $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^p}, \quad p>1$

1.4 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}3^{k-1}}$

1.5 $\displaystyle\sum_{k=1}^{\infty} \left(10+\frac{1}{k}\right) k^{-e}$

1.6 $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2-\sqrt{k}}{k^4-k^2+1}$

2. Prove that each of the following series diverges.

2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$

2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\ln^p(k+1)}, \quad p>0$

2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+2k+3}{k^3-2k^2+\sqrt{2}}$

2.4 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\ln^p k}, \quad p\le 1$

3. Use the Comparison Test to determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\ln k}{k}}$ converges or diverges.

4. Find all $p \ge 0$ such that the following series converges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\ln^p(k+1)}$

5. If $a_k \ge 0$ is a bounded sequence, prove that $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$ converges for all $p>1$.

6. If $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges, prove that $\displaystyle\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$ converges for all $p \ge 0$. What happen if $p<0$?

7. Prove that if $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ coverge, then $\displaystyle\sum_{k=1}^{\infty} a_kb_k$ also converges.

8. Suppose tha $a, b \in \mathbb{R}$ satisfy $\frac{b}{a} \in \mathbb{R}\backslash\mathbb{Z}$. Find all $q>0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)q^k} \qquad \text{converges.}$$

9. Suppose that $a_k \to 0$. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\displaystyle\sum_{k=1}^{\infty} (a_{2k}+a_{2k+1})$ converges.

8.3 Absolute convergence

Theorem 8.3.1 (**Cauchy Criterion**) *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$*, there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \textit{imply} \quad \left|\sum_{k=n}^{m} a_k\right| < \varepsilon.$$

Proof. Let s_n represent the sequence of partial sum of $\sum_{k=1}^{\infty} a_k$ and set $s_0 = 0$. By the Cauchy's Theorem (Theorem 2.4.5), s_n converges if and only if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \text{imply} \quad |s_m - s_{n-1}| < \varepsilon.$$

For all $m > n \geq 1$, we obtain

$$\left|\sum_{k=n}^{m} a_k\right| = |s_m - s_{n-1}| < \varepsilon.$$

The proof is complete. □

Corollary 8.3.2 *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$*, there is an* $N \in \mathbb{N}$ *such that*

$$n \geq N \quad \textit{implies} \quad \left|\sum_{k=n}^{\infty} a_k\right| < \varepsilon.$$

Proof. Exercise. □

ABSOLUTE CONVERGENCE.

Definition 8.3.3 *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series.*

1. *S is said to* ***converge absolutely*** *if and only if* $\sum_{k=1}^{\infty} |a_k| < \infty$.
2. *S is said to* ***converge conditionally*** *if and only if S converges but not absolutely.*

Theorem 8.3.4 *A series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely if and only if for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \textit{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Proof. The Cauchy Criterion gives us the Theorem 8.3.4. □

Theorem 8.3.5 *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Then $\sum_{k=1}^{\infty} |a_k|$ converges.
Let $\varepsilon > 0$. By Theorem 8.3.4, there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Apply the Triangle Inequality, we obtain

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon.$$

By the Cauchy Criterion, we conclude that $\sum_{k=1}^{\infty} a_k$ converges. □

Example 8.3.6 *Prove that* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ *converges absolutely but* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *is not.*

Solution. We consider

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Since the first and second series are a p-series such that $p = 2$ and $p = 1$, respectively, we obtain the first series converges but the second series is not. We conclude that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely but $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is not.

LIMIT SUPREMUM.

Definition 8.3.7 *The supremum s of the set of adherent points of a sequence $\{x_k\}$ is called the* ***limit supremum*** *of $\{x_k\}$, denoted by $s := \limsup_{k\to\infty} x_k$, i.e.,*

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup\{x_k : k \geq n\}.$$

Example 8.3.8 *Evaluate limit supremum of the following sequences.*

1. $x_k = \frac{1}{k}$ *2.* $y_k = \frac{(-1)^k}{k}$ *3.* $z_k = 1 + (-1)^k$

Solution. By the Definition of limit supremum, we have

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup\left\{\frac{1}{k} : k \geq n\right\} = \lim_{n\to\infty} \sup\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\begin{aligned}\limsup_{k\to\infty} y_k &= \lim_{n\to\infty} \sup\left\{\frac{(-1)^k}{k} : k \geq n\right\} \\ &= \lim_{n\to\infty} \sup\begin{cases} \frac{1}{n}, -\frac{1}{n+1}, \frac{1}{n+2}, \ldots & \text{if } n \text{ is even} \\ -\frac{1}{n}, \frac{1}{n+1}, -\frac{1}{n+2}, \ldots & \text{if } n \text{ is odd}\end{cases} \\ &= \lim_{n\to\infty} \frac{1}{n} = 0\end{aligned}$$

$$\limsup_{k\to\infty} z_k = \lim_{n\to\infty} \sup\left\{(-1)^k + 1 : k \geq n\right\} = \lim_{n\to\infty} \sup\{0, 2\} = \lim_{n\to\infty} 2 = 2.$$

Theorem 8.3.9 *Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence.*

1. *If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.*
2. *If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many k.*

Proof. Let $x \in \mathbb{R}$ and $s := \limsup_{k\to\infty} x_k$.
1. Assume that $s < x$. Suppose to the contary that there exist natural numbers

$$k_1 < k_2 < k_3 < \cdots \quad \text{such that} \quad x_{k_j} \geq x \quad \text{for } j \in \mathbb{N}.$$

If $\{x_{k_j}\}$ is unbounded above, it implies that $\sup\{x_k : k \geq n\}$ is unbounded above so $s = \infty$, a contradiction. If $\{x_{k_j}\}$ is bounded above by C, then $x \leq x_{k_j} \leq C$ for all $j \in \mathbb{N}$. Thus, by the Bolzano-Weierstrass Theorem and the fact that $x \leq x_{k_j}$, $\{x_{k_j}\}$ has a convergent subsequence. It implies that $s > x$, another contradiction.
2. Assume that $s > x$. There is a $c \in \mathbb{R}$ such that $x < c < s$. By the Approximation Property in the Theorem 2.2.5, there is a subsequence $\{x_{k_j}\}$ that converges to c; i.e., $x_k > x$ for lagre j. □

Theorem 8.3.10 *Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence. If $x_k \to x$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} x_k = x.$$

Proof. Assume that $x_k \to x$ as $k \to \infty$. By the Theorem 2.1.18, any subsequence $\{x_{k_j}\}$ also converges to x. It implies that $\limsup_{k\to\infty} x_k = x$. □

Example 8.3.11 *Evaluate limit supremum of $\left\{\frac{k}{k+1}\right\}$.*

Solution. Since $\lim_{k\to\infty} \frac{k}{k+1} = 1$, we obtain by Theorem 8.3.10 that

$$\limsup_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k}{k+1} = 1.$$

ROOT TEST.

Theorem 8.3.12 (**Root Test**) *Let* $a_k \in \mathbb{R}$ *and* $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. *If* $r < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*
2. *If* $r > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Proof. 1. Assume that $r < 1$. Then there is an $x \in \mathbb{R}$ such that $r < x < 1$.
We notice that the geometric series $\sum_{k=1}^{\infty} x^k$ converges. By Theorem 8.3.9, we have

$$|a_k|^{\frac{1}{k}} < x \quad \text{for large } k.$$

It follows that $0 < |a_k| < x^k$ for large k. By the Comparison Test, $\sum_{k=1}^{\infty} |a_k|$ converges.
2. Assume that $r > 1$. By Theorem 8.3.9, we have

$$|a_k|^{\frac{1}{k}} > 1 \quad \text{for infinitely many } k.$$

It follows that $|a_k| > 1$ for infinitely many k. Then the limit of a_k is not zero.
By the Divergence Test, $\sum_{k=1}^{\infty} a_k$ diverges. □

Example 8.3.13 *Prove that* $\sum_{k=1}^{\infty} \left(\frac{k}{1+2k}\right)^k$ *converges absolutely.*

Solution. We notice that

$$\limsup_{k\to\infty} \left| \left(\frac{k}{1+2k}\right)^k \right|^{\frac{1}{k}} = \limsup_{k\to\infty} \frac{k}{1+2k} = \lim_{k\to\infty} \frac{k}{1+2k} = \frac{1}{2} < 1.$$

By the Root Test, we conclude that $\sum_{k=1}^{\infty} \left(\frac{k}{1+2k}\right)^k$ converges absolutely.

Example 8.3.14 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3+(-1)^k}{2}\right)^k$ *diverges.*

Solution. We notice that

$$\limsup_{k\to\infty}\left|\left(\frac{3+(-1)^k}{2}\right)^k\right|^{\frac{1}{k}} = \limsup_{k\to\infty}\left|\frac{3+(-1)^k}{2}\right|$$
$$= \lim_{n\to\infty}\sup\{1,2\} = \lim_{n\to\infty} 2 = 2 > 1.$$

By the Root Test, we conclude that $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3+(-1)^k}{2}\right)^k$ diverges.

RATIO TEST.

Theorem 8.3.15 (**Ratio Test**) *Let $a_k \in \mathbb{R}$ with $a_k \neq 0$ for large k and suppose that*

$$r := \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

exists as an extended real number.

1. *If $r < 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges absolutely.*
2. *If $r > 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. 1. Assume that $r < 1$. Then there is an $x \in \mathbb{R}$ such that $r < x < 1$.
We notice that the geometric series $\displaystyle\sum_{k=1}^{\infty} x^k$ converges.
By Theorem 8.3.10, we have $r = \displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$. By Theorem 8.3.9, we obtain

$$\left|\frac{a_{k+1}}{a_k}\right| < x \quad \text{for large } k.$$

It follows that $\displaystyle\left|\frac{a_{k+1}}{a_k}\right| < x = \frac{x^{k+1}}{x^k}$ for large k which is equivalent to

$$\frac{|a_{k+1}|}{x^{k+1}} < \frac{|a_k|}{x^k} \quad \text{for large } k.$$

Then $\displaystyle\frac{|a_k|}{x^k}$ is decreasing and bounded. So, there is an $M > 0$ such that $|a_k| \leq Mx^k$ for all $k \in \mathbb{N}$.
We see that $\displaystyle\sum_{k=1}^{\infty} Mx^k$ converges. By the Comparison Test, $\displaystyle\sum_{k=1}^{\infty}|a_k|$ converges.

2. Assume that $r > 1$. By Theorem 8.3.9, we have

$$\left|\frac{a_{k+1}}{a_k}\right| > 1 \quad \text{for infinitely many } k.$$

It follows that $|a_{k+1}| > |a_k|$ for infinitely many k. Thus, a_k is increasaing which induces nonzero limit of a_k. By the Divergence Test, $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges. □

Example 8.3.16 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$ *converges absolutely.*

Solution. We notice that

$$\lim_{k\to\infty}\left|\frac{3^{k+1}}{(k+1)!}\cdot\frac{k!}{3^k}\right| = \lim_{k\to\infty}\frac{3}{k+1} = 0 < 1.$$

By the Ratio Test, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$ converges.

Example 8.3.17 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ *diverges.*

Solution. We notice that

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(k+1)^{k+1}}{(k+1)!}\cdot\frac{k!}{k^k}\right| &= \lim_{k\to\infty}\frac{(k+1)^{k+1}}{(k+1)k^k} \\
&= \lim_{k\to\infty}\frac{(k+1)^k}{k^k} \\
&= \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k \\
&= \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e > 1
\end{aligned}$$

By the Ratio Test, we conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ diverges.

Exercises 8.3

1. Prove that each of the following series converges.

1.1 $\sum_{k=1}^{\infty} \frac{1}{k!}$ 1.2 $\sum_{k=1}^{\infty} \frac{1}{k^k}$ 1.3 $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ 1.4 $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

2. Decide, using results convered so far in this chapter, which of the following series converge and which diverge.

2.1 $\sum_{k=1}^{\infty} \frac{k^2}{\pi^k}$ 2.4 $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+3}\right)^k$ 2.7 $\sum_{k=1}^{\infty} \left(\frac{k!}{(k+2)!}\right)^{k^2}$

2.2 $\sum_{k=1}^{\infty} \frac{k!}{2^k}$ 2.5 $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$ 2.8 $\sum_{k=1}^{\infty} \left(\frac{3+(-1)^k}{3}\right)^k$

2.3 $\sum_{k=1}^{\infty} \frac{k!}{2^k+3^k}$ 2.6 $\sum_{k=1}^{\infty} \left(\pi - \frac{1}{k}\right) k^{-1}$ 2.9 $\sum_{k=1}^{\infty} \frac{(1+(-1)^k)^k}{e^k}$

3. Define a_k recursively by $a_1 = 1$ and

$$a_k = (-1)^k \left(1 + k\sin\left(\frac{1}{k}\right)\right)^{-1} a_{k-1}, \quad k > 1.$$

Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

4. Suppose that $a_k \geq 0$ and $\sqrt[k]{a_k} \to a$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k x^k$ converges absolutely for all $|x| < \frac{1}{a}$ if $a \neq 0$ and for all $x \in \mathbb{R}$ if $a = 0$.

5. For eachof the following, find all values of $p \in \mathbb{R}$ for which the given series converges absolutely.

5.1 $\sum_{k=2}^{\infty} \frac{1}{k \ln^p k}$ 5.3 $\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ 5.5 $\sum_{k=1}^{\infty} \frac{2^{kp} k!}{k^k}$

5.2 $\sum_{k=2}^{\infty} \frac{1}{\ln^p k}$ 5.4 $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k^p - 1)}$ 5.6 $\sum_{k=1}^{\infty} (\sqrt{k^{2p}+1} - k^p)$

6. Suppose that $a_{kj} \geq 0$ for $k, j \in \mathbb{N}$. Set

$$A_k = \sum_{j=1}^{\infty} a_{kj}$$

for each $k \in \mathbb{N}$, and suppose that $\sum_{k=1}^{\infty} A_k$ converges.

   6.1 Prove that $\displaystyle\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right)$

   6.2 Show that $\displaystyle\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right)$

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove that $\sum_{k=1}^{\infty} |a_k|^p$ converges for all $p \geq 1$.

8. Suppose that $\sum_{k=1}^{\infty} a_k$ converges conditionally. Prove that $\sum_{k=1}^{\infty} k^p a_k$ diverges for all $p \geq 1$.

9. Let $a_n > 0$ for $n \in \mathbb{N}$. Set $b_1 = 0$, $b_2 = \ln\left(\frac{a_2}{a_1}\right)$, and

$$b_k = \ln\left(\frac{a_k}{a_{k-1}}\right) - \ln\left(\frac{a_{k-1}}{a_{k-2}}\right), \quad k = 3, 4, ...$$

   9.1 Prove that $r = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ if exists and is positive, then

$$\lim_{n\to\infty} \ln(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sum_{k=1}^{\infty}\left(1 - \frac{k-1}{n}\right) b_k = \sum_{k=1}^{\infty} b_k = \ln r.$$

   9.2 Prove that if $a_n \in \mathbb{R}\backslash\{0\}$ and $\left|\frac{a_{n+1}}{a_n}\right| \to r$ as $n \to \infty$, for some $r > 0$, then $|a_n|^{\frac{1}{n}} \to r$ as $n \to \infty$.

8.4 Alternating series

Theorem 8.4.1 (**Abel's Formula**) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *and* $\{b_k\}_{k\in\mathbb{N}}$ *be real sequences, and for each pair of integers* $n \geq m \geq 1$ *set*

$$A_{n,m} := \sum_{k=m}^{n} a_k.$$

Then

$$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$$

for all integers $n > m \geq 1$.

Proof. Since $A_{k,m} - A_{k-1,m} = a_k$ for $k > m$ and $A_{m,m} = a_m$, we obtain

$$\begin{aligned}
\sum_{k=m}^{n} a_k b_k &= a_m b_m + \sum_{k=m+1}^{n} a_k b_k \\
&= a_m b_m + \sum_{k=m+1}^{n} (A_{k,m} - A_{k-1,m}) b_k \\
&= a_m b_m + \sum_{k=m+1}^{n} A_{k,m} b_k - \sum_{k=m+1}^{n} A_{k-1,m} b_k \\
&= a_m b_m + \sum_{k=m+1}^{n} A_{k,m} b_k - \sum_{k=m}^{n-1} A_{k,m} b_k \\
&= a_m b_m + \sum_{k=m+1}^{n-1} A_{k,m} b_k + A_{n,m} b_n - \sum_{k=m+1}^{n-1} A_{k,m} b_k - A_{m,m} b_{m+1} \\
&= A_{m,m} b_m + A_{n,m} b_n - A_{m,m} b_{m+1} - \sum_{k=m+1}^{n-1} A_{k,m}(b_{k+1} - b_k) \\
&= A_{n,m} b_n - A_{m,m}(b_{m+1} - b_m) - \sum_{k=m+1}^{n-1} A_{k,m}(b_{k+1} - b_k) \\
&= A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)
\end{aligned}$$

The proof is complete. □

Theorem 8.4.2 (**Dirichilet's Test**) *Let $\{a_k\}$ and $\{b_k\}$ be sequences in $\mathbb{R}$. If the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ is bounded and $b_k \downarrow 0$ as $k \to \infty$, then*

$$\sum_{k=1}^{n} a_k b_k \quad \textit{converges.}$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ be bounded. Assume that b_k is decreasing and converges to zero. There is an $M > 0$ such that

$$|s_n| = \left|\sum_{k=1}^{n} a_k\right| \le M \quad \text{for all } n \in \mathbb{N}.$$

By the triangle inequality, for $n > m > 1$.

$$|A_{n,m}| = \left|\sum_{k=m}^{n} a_k\right| = |s_n - s_{m-1}| \le |s_n| + |s_{m-1}| \le M + M = 2M.$$

Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that

$$k \ge N \quad \text{implies} \quad |b_k| < \frac{\varepsilon}{2M}.$$

Since b_k is decreasing and converges to zero, $b_k - b_{k+1} > 0$ and $b_k > 0$ for all $k \in \mathbb{N}$.

By Abel's Formula and telescoping, for $n > m \ge N$, we obtain

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k b_k\right| &= \left|A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)\right| \\
&\le |A_{n,m}||b_n| + \left|\sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)\right| \\
&\le 2M|b_n| + \sum_{k=m}^{n-1} |A_{k,m}||b_{k+1} - b_k| \\
&\le 2M b_n + \sum_{k=m}^{n-1} 2M(b_k - b_{k+1}) \\
&= 2M b_n + 2M(b_m - b_n) \\
&= 2M b_m < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

Thus, $\sum_{k=1}^{n} a_k b_k$ converges. □

Corollary 8.4.3 (**Alternating Series Test (AST)**) *If $a_k \downarrow 0$ as $k \to \infty$, then*

$$\sum_{k=1}^{\infty}(-1)^k a_k \quad \textit{converges.}$$

Moreover, if $\sum_{k=1}^{\infty} a_k$ *converges, then*

$$\sum_{k=1}^{\infty}(-1)^k a_k \quad \textit{converges conditionally.}$$

Proof. Since the partial sums of $\sum_{k=1}^{\infty}(-1)^k$ are bounded, $\sum_{k=1}^{\infty}(-1)^k a_k$ converges by Dirichilet's Test. □

Example 8.4.4 *Prove that* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally.*

Solution. If $a_k = \frac{1}{k}$, we see that a_k is decreasing and converges to 0. By AST, we have $\sum_{k=1}^{\infty}(-1)^k a_k$ converges. It is clear that $\sum_{k=1}^{\infty} |(-1)^k a_k| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-Series Test ($p = 1$).
We conclude that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally.

Example 8.4.5 *Prove that* $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ *converges conditionally.*

Solution. Let $a_k = \frac{1}{\ln k}$. Since $k + 1 > k > 0$, $\ln(k + 1) > \ln k$. It implies that

$$\frac{1}{\ln(k+1)} < \frac{1}{\ln k} \quad \text{for all } k > 1.$$

Then a_k is decreasing and converges to 0. By AST, we obtain $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converges.
By Example 8.2.11, $\sum_{k=2}^{\infty} \frac{1}{\ln k}$. We conclude that $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converges conditionally.

Example 8.4.6 *Prove that* $S(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ *converges for each* $x \in \mathbb{R}$.

Solution. Let $x \in \mathbb{R}$. If $x = 2\ell\pi$ where $\ell \in \mathbb{Z}$, then

$$\sum_{k=1}^{\infty} \frac{\sin(2k\ell\pi)}{k} = 0 < \infty.$$

For case $x \neq 2\ell\pi$ for all $\ell \in \mathbb{Z}$. It's easy to see that $\left\{\frac{1}{k}\right\}$ is decreasing and $\lim_{k\to\infty} \frac{1}{k} = 0$.
Define

$$S_n = \sum_{k=1}^{n} \sin(kx)$$

Use trigonometry properties and teleascoping, we have

$$\begin{aligned}
\left(2\sin\left(\frac{x}{2}\right)\right) S_n &= \left(2\sin\left(\frac{x}{2}\right)\right) \sum_{k=1}^{n} \sin(kx) \\
&= \sum_{k=1}^{n} 2\sin(kx)\sin\left(\frac{x}{2}\right) \\
&= \sum_{k=1}^{n} \left[\cos\left(kx - \frac{x}{2}\right) - \cos\left(kx + \frac{x}{2}\right)\right] \\
&= \sum_{k=1}^{n} \left[\cos x\left(k - \frac{1}{2}\right) - \cos x\left(k + \frac{1}{2}\right)\right] \\
&= \cos x\left(\frac{1}{2}\right) - \cos x\left(n + \frac{1}{2}\right).
\end{aligned}$$

Since $\sin\left(\frac{x}{2}\right) \neq 0$ for all $x \neq 2\ell\pi$. We obatin

$$\begin{aligned}
\left|\left(2\sin\left(\frac{x}{2}\right)\right) S_n\right| &= \left|\cos x\left(\frac{1}{2}\right) - \cos x\left(n + \frac{1}{2}\right)\right| \\
\left|\left(2\sin\left(\frac{x}{2}\right)\right)\right| |S_n| &= \left|\cos x\left(\frac{1}{2}\right)\right| + \left|\cos x\left(n + \frac{1}{2}\right)\right| + \leq 1 + 1 = 2 \\
|S_n| &\leq \frac{1}{\left|\left(\sin\left(\frac{x}{2}\right)\right)\right|} = \left|\csc\left(\frac{x}{2}\right)\right|.
\end{aligned}$$

So, S_n is bounded for each $x \neq 2\ell\pi$. By Dirichilet's Test, it implies that

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \text{ converges for all } x \neq 2\ell\pi.$$

Therefore, we conclude that

$$S(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \text{ converges for all } x \in \mathbb{R}.$$

Exercises 8.4

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left(\frac{\pi}{2}-\arctan k\right)$

1.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{2^k}$

1.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}$

1.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^p}, \quad p>0$

1.5 $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^p}, \quad x\in\mathbb{R},\ p>0$

1.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\frac{2\cdot 4\cdots(2k)}{1\cdot 3\cdots(2k-1)}$

1.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\ln(e^k+1)}$

1.8 $\displaystyle\sum_{k=1}^{\infty}\frac{\arctan k}{4k^3-1}$

2. For each of the following, find all values $x\in\mathbb{R}$ for which the given series converges.

2.1 $\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$

2.2 $\displaystyle\sum_{k=1}^{\infty}\frac{x^{3k}}{2^k}$

2.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{\sqrt{k^2+1}}$

2.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(x+2)^k}{k\sqrt{k+1}}$

2.5 $\displaystyle\sum_{k=1}^{\infty}\frac{2^k(x+1)^k}{k!}$

2.6 $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k(x+3)}{\cos k}\right)^k$

3. Using any test coveredin this chapter, find out which of the following series converge absolutely, which converge conditionally, and which diverge.

3.1 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^3}{(k+1)!}$

3.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)(-3)\cdots(1-2k)}{1\cdot 4\cdots(3k-2)}$

3.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(k+1)^k}{p^k k!}, \quad p>e$

3.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k+1}$

3.5 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k+1}}{\sqrt{k}k^k}$

3.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sin k}{k!}$

3.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}$

3.8 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\ln(k+2)}{k}$

4. **ABEL'S TEST.** Suppose that $\sum_{k=1}^{\infty} a_k$ converges and $b_k \downarrow b$ as $k \to \infty$. Prove that

$$\sum_{k=1}^{\infty} a_k b_k \quad \text{converges.}$$

5. Use Dirichilet's Test to prove that

$$S(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

   converges for all $x \in \mathbb{R}$.

6. Prove that $\sum_{k=1}^{\infty} a_k \cos(kx)$ converges for every $x \in (0, 2\pi)$ and every $a_k \downarrow 0$.
   What happens when $x = 0$?

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Prove that if $b_k \uparrow \infty$ and $\sum_{k=1}^{\infty} a_k b_k$ converges, then

$$b_m \sum_{k=m}^{\infty} a_k \to 0 \quad \text{as} \quad m \to \infty.$$

Index

Reference

Gerald B. Folland. (1999). **Real Analysis Modern Technique and Their Applications**. John Wiley & Sons, Inc., New York.

Halsey L. Royden and Prtrick M. Fitzpatrick. (2010). **Real Analysis** (Fourth Edition). Pearson Education, Inc. New Jersey.

Pual Glendinning. (2012). **Maths in minutes**. Quercus Editions Ltd, London, England.

William R. Wade. (2004). **An Introduction Analysis** (Third Edition). Pearson Education. Inc., New Jersey.

Vita

Assistant Professor Thanatyod Jampawai, Ph.D.

- B.Sc. (Mathematics, 2nd class honours), Chulalongkorn University, 2006
- M.Sc. (Mathematics), Chulalongkorn University, 2009
- Ph.D. (Mathematics), Chulalongkorn University, 2014
- Assistant Professor of Mathematics, Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University.

Contact me

Email: thanatyod.ja@ssru.ac.th

Website: www.eledu.ssru.ac.th/thanatyod_ja

Office: 1144