Math SSRU

Solution Assignment 10
MAC3309 Mathematical Analysis

Topic	Reimann sum and the Fundametal Theorem of Calculus	**Score**	10 marks
Time	12*th* Week		
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1. Let $f(x) = 2x^2 + 1$ where $x \in [0,1]$ and $P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$ be a partition of $[0,1]$. Show that if $f(t_i)$ is choosen by the **right end point** and **left end point** in each subinterval, then two $I(f)$, depend on two methods, are **NOT** different.

Solution. The Right End Point : Choose $f(t_j) = f(\frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[2\left(\frac{j}{n}\right)^2 + 1\right] \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{2}{n^2}\cdot j^2 + 1\right] = \frac{1}{n}\left[\frac{2}{n^2}\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 1\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + n\right] = \frac{(n+1)(2n+1)}{3n^2} + 1
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2} + 1 = \frac{2}{3} + 1 = \frac{5}{3} \quad \#$$

The Left End Point : Choose $f(t_j) = f(\frac{j-1}{n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j-1}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[2\left(\frac{j-1}{n}\right)^2 + 1\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}\sum_{j=1}^{n}(j-1)^2 + \sum_{j=1}^{n} 1\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}[0^2 + 1^2 + 2^2 + \cdots (n-1)^2] + n\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}\cdot\frac{(n-1)(n)(2(n-1)+1)}{6} + n\right] = \frac{(n-1)(2n-1)}{3n^2} + 1
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{3n^2} + 1 = \frac{2}{3} + 1 = \frac{5}{3} \quad \#$$

2. Le $f(x) = 2x^2 + 1$ where $x \in [0,1]$ and $P = \left\{ \frac{j}{2^n} : j = 0, 1, ..., 2^n \right\}$ be a partition of $[0,1]$. Show that if $f(t_i)$ is choosen by the **right end point** and **left end point** in each subinterval, then two $I(f)$, depend on two methods, are **NOT** different.

Solution. The Right End Point : Choose $f(t_j) = f(\frac{j}{2^n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{2^n}$ for all $j = 1, 2, 3, ..., 2^n$. We obtain

$$\begin{aligned}\sum_{j=1}^{2^n} f(t_j)\Delta x_j &= \sum_{j=1}^{2^n} f\left(\frac{j}{2^n}\right)\frac{1}{2^n} = \frac{1}{2^n}\sum_{j=1}^{2^n}\left[2\left(\frac{j}{2^n}\right)^2 + 1\right] \\ &= \frac{1}{2^n}\sum_{j=1}^{2^n}\left[\frac{2}{2^{2n}}\cdot j^2 + 1\right] = \frac{1}{2^n}\left[\frac{2}{2^{2n}}\sum_{j=1}^{2^n} j^2 + \sum_{j=1}^{2^n} 1\right] \\ &= \frac{1}{2^n}\left[\frac{2}{2^{2n}}\cdot\frac{2^n(2^n+1)(2\cdot 2^n+1)}{6} + 2^n\right] = \frac{(2^n+1)(2\cdot 2^n+1)}{3\cdot 2^n} + 1\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(2^n+1)(2\cdot 2^n+1)}{3\cdot 2^n} + 1 = \frac{2}{3} + 1 = \frac{5}{3} \quad \#$$

The Left End Point : Choose $f(t_j) = f(\frac{j-1}{2^n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., 2^n$. We obtain

$$\begin{aligned}\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{2^n} f\left(\frac{j-1}{2^n}\right)\frac{1}{2^n} = \frac{1}{2^n}\sum_{j=1}^{2^n}\left[2\left(\frac{j-1}{2^n}\right)^2 + 1\right] \\ &= \frac{1}{2^n}\left[\frac{2}{2^{2n}}\sum_{j=1}^{2^n}(j-1)^2 + \sum_{j=1}^{2^n} 1\right] \\ &= \frac{1}{2^n}\left[\frac{2}{2^{2n}}[0^2 + 1^2 + 2^2 + \cdots (2^n-1)^2] + n\right] \\ &= \frac{1}{2^n}\left[\frac{2}{2^{2n}}\cdot\frac{(2^n-1)(2^n)(2(2^n-1)+1)}{6} + 2^n\right] = \frac{(2^n-1)(2\cdot 2^n-1)}{3\cdot 2^{2n}} + 1\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(2^n-1)(2\cdot 2^n-1)}{3\cdot 2^{2n}} + 1 = \frac{2}{3} + 1 = \frac{5}{3} \quad \#$$

3. Suppose that f is integrable on $[a,b]$ and $\alpha \in \mathbb{R}$. Prove that αf is also integrable on $[a,b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx.$$

Proof. Assume that f is integrable on $[a,b]$ and $\alpha \in \mathbb{R}$.
Let $\varepsilon > 0$. There is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that

$$\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{|\alpha|+1}$$

where $t_j \in [x_{j-1}, x_j]$ and $\Delta x_j = x_j - x_{j-1}$. Thus,

$$\begin{aligned}\left|\sum_{j=1}^{n}\alpha f(t_j)\Delta x_j - \alpha\int_a^b f(x)\,dx\right| &= |\alpha|\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx\right| \\ &< |\alpha|\cdot\frac{\varepsilon}{|\alpha|+1} = \left(\frac{|\alpha|}{|\alpha|+1}\right)\varepsilon < 1\cdot\varepsilon = \varepsilon\end{aligned}$$

Therefore, αf is integrable on $[a,b]$ and $\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx$. □

4. Use the Reimann sum to prove that

$$\int_a^b \alpha \, dx = \alpha(b-a).$$

Proof. Let $f(x) = \alpha$ where $x \in [a,b]$ and $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$. Let t_j be in subinterval $[x_{j-1}, x_j]$. Then

$$\begin{aligned} I(f) &= \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha[x_j - x_{j-1}] \\ &= \lim_{n\to\infty} \alpha[x_n - x_0] = \alpha(b-a) \end{aligned}$$

Thus, $\displaystyle\int_a^b \alpha \, dx = \alpha(b-a)$. □

5. Use the chain rule and the first fundamental of calculus to prove that

$$\frac{d}{dx}\int_a^{u(x)} f(t)\, dt = f(u(x)) \cdot u'(x),$$

where a is a constant.

Proof. Define $F(x) = \displaystyle\int_a^x f(t)\, dt$. By the first fundamental of calculus, $F'(x) = f(x)$. Apply the chain rule,

$$\begin{aligned} \frac{d}{dx}F(u(x)) &= F'(u(x)) \cdot u'(x) \\ \frac{d}{dx}\int_a^{u(x)} f(t)\, dt &= f(u(x)) \cdot u'(x) \end{aligned}$$

□

6. Use the first fundamental of calculus to find $F(0)$ and $F'(0)$. Define

$$F(x) = \int_{e^{-x}}^{e^x} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt \quad \text{where } x \geq 0.$$

Solution. It is easy to see that

$$F(0) = \int_{e^{-0}}^{e^0} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt = \int_1^1 \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt = 0. \quad \#$$

Consider

$$\begin{aligned} F(x) &= \int_{e^{-x}}^{0} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt + \int_0^{e^x} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt \\ &= -\int_0^{e^{-x}} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt + \int_0^{e^x} \frac{1 + t \cdot f(t)}{1 + f(t)}\, dt \end{aligned}$$

Apply the first fundamental theorem of calculus,

$$\begin{aligned} F'(x) &= -\frac{1 + e^{-x} \cdot f(e^{-x})}{1 + f(e^{-x})} \cdot (-e^{-x}) + \frac{1 + e^x \cdot f(e^x)}{1 + f(e^x)} \cdot e^x \\ F'(0) &= \frac{1 + 1 \cdot f(1)}{1 + f(1)} \cdot 1 + \frac{1 + 1 \cdot f(1)}{1 + f(1)} \cdot 1 = 2 \quad \# \end{aligned}$$

7. Assume that f is differentiable on $(0,1)$ and integrable on $[0,1]$. Show that

$$\int_0^1 xf'(x) + f(x)\,dx = f(1).$$

Solution. It easy to see that $(xf(x))' = xf'(x) + f(x)$. Apply the fundamental theorem of calculus,

$$\int_0^1 xf'(x) + f(x)\,dx = \int_0^1 (xf(x))'\,dx = 1f(1) - 0f(0) = f(1).$$

8. Let $u(x)$ be a real function on $[0,1]$ and be diferentiable on $(0,1)$.
Assume that $u(x) \neq 0$ for all $x \in [0,1]$ and $u(0) = u(1)$. Show that

$$\int_0^1 \frac{u'(x)}{u(x)}\,dx = 0.$$

Solution. Let $F(x) = \ln(|u(x)|)$ and $f(x) = \dfrac{u'(x)}{u(x)}$. Then

$$F'(x) = \frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)} = f(x)$$

Apply the Second Fundamental Thorem of Calculus,

$$\begin{aligned}\int_0^1 \frac{u'(x)}{u(x)}\,dx &= \int_0^1 f(x)\,dx = F(1) - F(0)\\ &= \ln(|u(1)|) - \ln(|u(0)|)\\ &= \ln\left|\frac{u(1)}{u(0)}\right| = \ln 1 = 0\end{aligned}$$