Math SSRU

Solution Assignment 11
MAC3309 Mathematical Analysis

Topic Integration by part and Infinite series **Score** 10 marks
Time 13*th* Week
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1. Let $f(x) = \int_0^{x^2} \sec^2(t^2)dt$. Use **integration by part** to show that

$$2\int_0^1 \sec^2(x^2)\,dx - 4\int_0^1 xf(x)\,dx = \tan 1.$$

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = \sec^2((x^2)^2)\cdot(x^2)' = 2x\sec^2(x^4).$$

We obtain

$$\begin{aligned}
4\int_0^1 xf(x)dx &= 2\int_0^1 (2x)f(x)dx = 2\int_0^1 (x^2)'f(x)dx \\
&= 2\left(\left[x^2 f(x)\right]_0^1 - \int_0^1 x^2 f'(x)dx\right) \\
&= 2\left(1f(1) - 0f(0) - \int_0^1 x^2\cdot 2x\sec^2(x^4)dx\right) \\
&= 2f(1) - \int_0^1 4x^3\sec^2(x^4)dx \\
&= 2\int_0^1 e^{x^2}dx - \int_0^1 (\tan x^4)'dx \\
&= 2\int_0^1 e^{x^2}dx - \left[\tan(x^4)\right]_0^1 = 2\int_0^1 \sec^2(x^2)dx - \tan 1
\end{aligned}$$

Thus,

$$2\int_0^1 \sec^2(x^2)\,dx - 4\int_0^1 xf(x)\,dx = \tan 1.$$

2. Let $f : [-a, a] \to \mathbb{R}$ where $a > 0$. Suppose that $f(-x) = f(x)$ for all $x \in [-a, a]$. Show that

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx.$$

Hint: By dividing the integral into two parts and changing of variable.

Solution. Consider

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx.$$

Next, we rewrite $\int_{-a}^0 f(x)dx$ by changing of variable $\phi(x) = -x$. From $f(-x) = f(x)$, we obatin

$$\begin{aligned}
\int_{-a}^0 f(x)dx &= \int_{-a}^0 f(-x)dx = -\int_{-a}^0 f(-x)(-x)'dx \\
&= -\int_{-a}^0 f(\phi(x))\phi'(x)dx = -\int_{\phi(-a)}^{\phi(0)} f(t)dt \\
&= -\int_a^0 f(x)dx = \int_0^a f(x)dx
\end{aligned}$$

Hence,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

3. Let $\int_{-1}^{0} f(t)\,dt = 2022$. Estimate the integral

$$\int_0^1 \frac{f\left(\frac{x-1}{x+1}\right)}{(x+1)^2}\,dx$$

Solution. Let $\phi(x) = \dfrac{x-1}{x+1} = 1 - \dfrac{2}{x+1}$. Then, $\phi(0) = -1$, $\phi(1) = 0$ and

$$\phi'(x) = \frac{2}{(x+1)^2}.$$

We obtain

$$\begin{aligned}
\int_0^1 \frac{f\left(\frac{x-1}{x+1}\right)}{(x+1)^2}\,dx &= \frac{1}{2}\int_0^1 f\left(\frac{x-1}{x+1}\right)\cdot\frac{2}{(x+1)^2}\,dx \\
&= \frac{1}{2}\int_0^1 f\left(\phi(x)\right)\cdot\phi'(x)\,dx \\
&= \frac{1}{2}\int_{\phi(0)}^{\phi(1)} f(t)\,dt \\
&= \frac{1}{2}\int_{-1}^{0} f(t)\,dt = \frac{1}{2}\cdot 2022 = 1011 \quad \#
\end{aligned}$$

4. Show that $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right)$ converges and find its value.

Solution. We will rewrite the infinite sum in term of telescoping series:

$$\begin{aligned}
\sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right) &= \sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right) = \sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\cdot\frac{k+2}{k+1}\right) \\
&= \sum_{k=1}^{\infty}\left[\ln\left(\frac{k}{k+1}\right) - \ln\left(\frac{k+1}{k+2}\right)\right] \\
&= \ln\left(\frac{1}{2}\right) - \lim_{k\to\infty}\ln\left(\frac{k+1}{k+2}\right) \\
&= -\ln 2 - \ln 1 = -\ln 2.
\end{aligned}$$

5. Use Telescoping Seires to show that Guass' formula : $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof. By the that $(k+1)^2 - k^2 = 2k+1$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n}(2k+1) &= \sum_{k=1}^{n}[(k+1)^2 - k^2] \\
2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 &= (n+1)^2 - 1^2 = n^2 + 2n \\
2\sum_{k=1}^{n} k + n &= n^2 + 2n \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}
\end{aligned}$$

□

6. Use Telescoping Seires to show that

$$\sum_{k=1}^{\infty} \frac{k-1}{2^k}$$

converges and find its values.

Solution. Consider the difference of

$$\frac{k-1}{2^k} = \frac{2k-k-1}{2^k} = \frac{2k}{2^k} - \frac{k+1}{2^k} = \frac{k}{2^{k-1}} - \frac{k+1}{2^k}.$$

By Telescopic series, we have

$$\sum_{k=1}^{\infty} \frac{k-1}{2^k} = \sum_{k=1}^{\infty} \left[\frac{k}{2^{k-1}} - \frac{k+1}{2^k}\right] = \frac{1}{2^0} - \lim_{k\to\infty} \frac{k+1}{2^k} = 1 - 0 = 1$$

7. Find all $x \in \mathbb{R}$ for which

$$\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1})$$

converges. For each such x, find the value of this series.

Solution. Consider

$$\begin{aligned} S_n &= \sum_{k=1}^{n} 3(x^k - x^{k-1})(x^k + x^{k-1}) \\ &= \sum_{k=1}^{\infty} 3(x^{2k} - x^{2k-2}) \\ &= 3[(x^2 - x^0) + (x^4 - x^2) + (x^6 - x^4) + \cdots + (x^{2n} - x^{2n-2})] \\ &= 3(x^{2n} - 1) \end{aligned}$$

Case $x = \pm 1$. Then $x^{2n} = 1$. So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3(1-1) = 0$$

Case $|x| < 1$. Then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3(x^{2n} - 1) = 3(0-1) = -3$$

Case $|x| > 1$. Then $\lim_{n\to\infty} S_n$ go to infinity.

Hence, $\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1})$ conveges if and only if $|x| \le 1$ and

$$\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1}) = \begin{cases} 0 & \text{if } x = \pm 1 \\ -3 & \text{if } |x| < 1 \\ \infty & \text{if } |x| > 1 \end{cases}.$$

8. Let π be a Pi constant. Show that

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right]$$

converges and find its value.

Solution. We rewrite the term of this series

$$\begin{aligned} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right] &= \frac{1}{\pi^{k^2}} - \frac{1}{\pi^{k^2-2k+1}} + \frac{1}{\pi^{k^2}} \cdot \frac{\pi^{k^2}}{\pi^k} \\ &= \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \left(\frac{1}{\pi} \right)^k \end{aligned}$$

Then, the first term is telescoping series and the second term is geometric series. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right] &= \sum_{k=1}^{\infty} \left(\frac{1}{\pi^{k^2}} - \frac{1}{\pi^{(k-1)^2}} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi} \right)^k \\ &= -\sum_{k=1}^{\infty} \left(\frac{1}{\pi^{(k-1)^2}} - \frac{1}{\pi^{k^2}} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{\pi} \right)^k \\ &= -1 + \lim_{k \to \infty} \frac{1}{\pi^{k^2}} + \frac{\frac{1}{\pi}}{1 - \frac{1}{\pi}} \\ &= -1 + 0 + \frac{1}{\pi - 1} = \frac{2 - \pi}{\pi - 1} \qquad \# \end{aligned}$$