Solution Assignment 12
MAC3309 Mathematical Analysis

Topic Test of Seires **Score** 10 marks
Time $14th$ Week
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1. Dertermine whether the following series are convergent.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$

   Solution. Since $1 \leq k$ for all $k \in \mathbb{N}$,
   $$1 \leq \sqrt[k]{k}.$$
   We obtain
   $$0 \leq \frac{1}{k} \leq \frac{\sqrt[k]{k}}{k} \quad \text{for all } k \geq 1.$$
   It's easy to see that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges because it is a p-series such that $p = 1$.
   By the Comparision Test, it implies that
   $$\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k} \quad \text{diverges.}$$

   (b) $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) k^{-\pi}$

   Solution. Consider
   $$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) k^{-\pi} = \sum_{k=1}^{\infty} \frac{1}{k^{\pi}} + \sum_{k=1}^{\infty} \frac{1}{k^{1+\pi}}.$$
   Then, two parts are p-series such that $p = \pi > 1$ and $p = 1 + \pi > 1$. So, each part of the series is converges. We conclude that
   $$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) k^{-\pi} \quad \text{converges.}$$

2. Find all $p \in \mathbb{R}$ such that the following series are convergent.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$ **Hint**: Use the Integral Test.

   Solution. Let $f(x) = \dfrac{\ln x}{x^p}$ where $x \geq 1$. First, we consider the derivative of the function:
   $$\begin{aligned} f'(x) &= \frac{x^p \cdot \frac{1}{x} - \ln x \cdot x^{p-1}}{x^{2p}} \\ &= \frac{x^{p-1}(1 - \ln x)}{x^{2p}} < 0 \qquad \text{if } x \geq 3 \end{aligned}$$
   So, f is decreasing on $[3, \infty)$. Next, we find p satisfying $\displaystyle\int_3^{\infty} f(x)dx < \infty$, i.e.,

$$\begin{aligned}
\int_3^\infty f(x)dx &= \int_3^\infty \frac{\ln x}{x^p} dx = \lim_{t\to\infty} \int_3^t (x^{-p}) \ln x\, dx \\
&= \lim_{t\to\infty} \int_3^t \left[\left(\frac{x^{-p+1}}{1-p} \right)' \ln x \right] dx \\
&= \lim_{t\to\infty} \left(\left[\frac{x^{-p+1}}{1-p} \ln x \right]_3^t - \int_3^t \left(\frac{x^{-p+1}}{1-p} \right) (\ln x)'\, dx \right) \\
&= \lim_{t\to\infty} \left(\left[\frac{t^{-p+1}}{1-p} \ln t - \frac{3^{-p+1}}{1-p} \ln 3 \right] - \int_3^t \left(\frac{x^{-p+1}}{1-p} \right) \frac{1}{x}\, dx \right) \\
&= \lim_{t\to\infty} \left(\frac{\ln t}{(1-p)t^{p-1}} - \frac{\ln 3}{(1-p)3^{p-1}} - \int_3^t \frac{x^{-p}}{1-p} dx \right) \\
&= \lim_{t\to\infty} \left(\frac{\ln t}{(1-p)t^{p-1}} - \frac{\ln 3}{(1-p)3^{p-1}} - \left[\frac{x^{-p+1}}{(1-p)^2} \right]_3^t \right) \\
&= \lim_{t\to\infty} \left(\frac{\ln t}{(1-p)t^{p-1}} - \frac{\ln 3}{(1-p)3^{p-1}} - \frac{1}{t^{p-1}(1-p)^2} + \frac{1}{3^{p-1}(1-p)^2} \right) \\
&= -\frac{\ln 3}{(1-p)3^{p-1}} + \frac{1}{3^{p-1}(1-p)^2} \qquad \text{if } p > 1
\end{aligned}$$

By the Integral Test, we conclude that

$$\sum_{k=2}^\infty \frac{\ln k}{k^p} \quad \text{if and only if} \quad p > 1.$$

(b) $\displaystyle\sum_{k=2}^\infty \frac{1}{k(\ln k)^p}$ **Hint**: Use the Integral Test

Solution. Let $f(x) = \dfrac{1}{x(\ln x)^p}$ where $x \geq 2$. First, we consider the derivative of the function:

$$\begin{aligned}
f'(x) &= -1(x(\ln x)^p)^{-2} \left((\ln x)^p + px(\ln x)^{p-1} \frac{1}{x} \right) \\
&= -\frac{1}{(x(\ln x)^p)^2} \left((\ln x)^p + p(\ln x)^{p-1} \right) \\
&= -\frac{(\ln x)^p}{(x(\ln x)^p)^2} \left(1 + \frac{p}{\ln x} \right) \qquad \text{......}(*)
\end{aligned}$$

Next, we find p satisfying $\int_2^\infty f(x)dx < \infty$, i.e.,

$$\begin{aligned}
\int_2^\infty f(x)dx &= \int_2^\infty \frac{1}{x(\ln x)^p} dx \\
&= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^p} dx \\
&= \lim_{t\to\infty} \left[(\ln x)^{-p+1} \right]_2^t \\
&= \lim_{t\to\infty} \left[\frac{1}{(\ln t)^{p-1}} - \frac{1}{(\ln 2)^{p-1}} \right] \\
&= -\frac{1}{(\ln 2)^{p-1}} < \infty \qquad \text{if } p > 1
\end{aligned}$$

Recheck in $(*)$, we get $f'(x) < 0$ if $p > 1$. So, f is decreasing when $x \geq 2$.
By the Integral Test, we conclude that

$$\sum_{k=2}^\infty \frac{1}{k(\ln k)^p} \quad \text{if and only if} \quad p > 1.$$

3. Prove that

$$\text{if } \sum_{k=1}^{\infty} |a_k| \text{ converges, then } \sum_{k=1}^{\infty} \frac{|a_k|}{k^p} \text{ converges for all } p > 0.$$

Hint: Use The Limit Comparision Test.

Proof. Let $p > 0$. Assume that $\sum_{k=1}^{\infty} |a_k|$ converges. Consider

$$\lim_{k\to\infty} \frac{\frac{|a_k|}{k^p}}{|a_k|} = \lim_{k\to\infty} \frac{1}{k^p} = 0$$

By the Limit Comparision Test, we conclude that $\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$. □

4. Use the **Limit Comparision Test** to show that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^p}\right) \quad \text{converges} \quad \text{if } p > 1.$$

Solution. Let $p > 1$ and $a_k = \arctan\left(\frac{1}{k^p}\right)$. Choose $b_k = \frac{1}{k^p}$.

Then the series $\sum_{k=1}^{\infty} b_k$ converges because it is a $p-$series such that $p > 1$. We obtain

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{\arctan\left(\frac{1}{k^p}\right)}{\frac{1}{k^p}} && \text{I.F. } \frac{0}{0} \\
&= \lim_{k\to\infty} \frac{\frac{1}{1+\left(\frac{1}{k^p}\right)^2} \cdot (-pk^{-p-1})}{-pk^{-p-1}} && \text{L'Hospital's Rule} \\
&= \lim_{k\to\infty} \frac{1}{1+\left(\frac{1}{k^p}\right)^2} = 1 > 0
\end{aligned}$$

By the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^p}\right) \quad \text{converges.}$$