Math SSRU

Solution Assignment 13
MAC3309 Mathematical Analysis

Topic	Absolute convergent and Alternating series	**Score**	10 marks
Time	15*th* Week		
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1. Use the **Root Test** to find all of $x \in \mathbb{R}$ such that

$$\sum_{k=1}^{\infty}\left(\frac{(kx+1)^2}{k^2+1}\right)^k \qquad \text{converges.}$$

Solution. Let $a_k = \left(\frac{(kx+1)^2}{k^2+1}\right)^k$. We consider

$$\begin{aligned}
\limsup_{k\to\infty}(|a_k|)^{\frac{1}{k}} &= \limsup_{k\to\infty}\left(\left|\left(\frac{(kx+1)^2}{k^2+1}\right)^k\right|\right)^{\frac{1}{k}}\\
&= \limsup_{k\to\infty}\left|\left(\frac{x^2k^2+2kx+1}{k^2+1}\right)\right|\\
&= \lim_{n\to\infty}\sup_{k\geq n}\left|\left(\frac{x^2k^2+2kx+1}{k^2+1}\right)\right|\\
&= \lim_{n\to\infty}\left|\left(\frac{x^2n^2+2nx+1}{n^2+1}\right)\right|\\
&= x^2
\end{aligned}$$

By the Root Test, the series converges if $x^2 < 1$. Then $|x| < 1$ or $x \in (-1, 1)$.
Therefore,

$$\sum_{k=1}^{\infty}\left(\frac{(kx+1)^2}{k^2+1}\right)^k \text{ converges if and only if } |x| < 1. \qquad \#$$

2. Use the **Ratio Test** to find all of $x \in \mathbb{R}$ such that Bessel function of first order $J_1(x)$ **converges** where

$$J_1(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k+1}}.$$

Solution. Let $a_k = \frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k+1}}$. Then

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+3}}{(k+1)!(k+2)!2^{2k+3}}\cdot\frac{k!(k+1)!2^{2k+1}}{(-1)^k x^{2k+1}}\right|\\
&= \lim_{k\to\infty}\left|\frac{(-1)x^2}{(k+1)(k+2)2^2}\right|\\
&= x^2\lim_{k\to\infty}\frac{1}{4(k+1)(k+2)}\\
&= x^2\cdot 0 = 0 < 1
\end{aligned}$$

By the Ratio Test, we conclude that $J_1(x)$ converges for all $x \in \mathbb{R}$.

3. Dethermine whether the following series are absolutely convergent or NOT.

(a) $\sum_{k=1}^{\infty}\left(\frac{k+1}{k+2}\right)^{k^2}$

Solution. Consider

$$\begin{aligned}\limsup_{k\to\infty}\left(\left|\left(\frac{k+1}{k+2}\right)^{k^2}\right|\right)^{\frac{1}{k}} &= \limsup_{k\to\infty}\left|\left(\frac{k+1}{k+2}\right)^{k}\right| \\ &= \lim_{n\to\infty}\sup_{k\geq n}\left|\left(\frac{k+1}{k+2}\right)^{k}\right| \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{n}\end{aligned}$$

Then,

$$\begin{aligned}L &= \left(\frac{n+1}{n+2}\right)^{n} \\ \ln L &= n\ln\left(\frac{n+1}{n+2}\right) \\ \lim_{n\to\infty}\ln L &= \lim_{n\to\infty} n\ln\left(\frac{n+1}{n+2}\right) \\ &= \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n+2}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{n+2}{n+1}\cdot\left(\frac{n+1}{n+2}\right)'}{-\frac{1}{n^2}} \\ &= \lim_{n\to\infty}\frac{\frac{n+2}{n+1}\cdot\frac{1}{(n+2)^2}}{-\frac{1}{n^2}} \\ &= \lim_{n\to\infty}\frac{-n^2}{(n+1)(n+2)} = -1 \\ \lim_{n\to\infty} L &= e^{-1}\end{aligned}$$

So,

$$\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{n} = e^{-1} < 1$$

By the Root Test, we conclude that

$$\sum_{k=1}^{\infty}\left(\frac{k+1}{k+2}\right)^{k^2} \text{ is absolutely convergent.}$$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(1+(-1)^k)^k}{e^k}$

Solution. Consider

$$\begin{aligned}
\limsup_{k\to\infty} \left(\left| \frac{(1+(-1)^k)^k}{e^k} \right| \right)^{\frac{1}{k}} &= \limsup_{k\to\infty} \frac{|1+(-1)^k|}{e} \\
&= \lim_{n\to\infty} \sup_{k\geq n} \frac{|1+(-1)^k|}{e} \\
&= \lim_{n\to\infty} \sup_{k\geq n} \left\{ \frac{|1+(-1)^n|}{e}, \frac{|1+(-1)^{n+1}|}{e}, \frac{|1+(-1)^{n+2}|}{e}, \dots \right\} \\
&= \lim_{n\to\infty} \sup_{k\geq n} \left\{ 0, \frac{2}{e} \right\} \\
&= \lim_{n\to\infty} \frac{2}{e} \\
&= \frac{2}{e} < 1
\end{aligned}$$

By the Root Test, we conclude that

$$\sum_{k=1}^{\infty} \frac{(1+(-1)^k)^k}{e^k} \text{ is absolutely convergent.}$$

4. Dethermine whether the following series are conditionally convergent or NOT.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$

Solution. Consider

$$\sum_{k=2}^{\infty} \left| \frac{(-1)^k}{k\ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{k\ln k}.$$

Let $f(x) = \dfrac{1}{x\ln x}$ where $x \geq 2$. Then

$$f'(x) = -(x\ln x)^2 (\ln x + 1) = -\frac{(1+\ln x)}{(x\ln x)^2} < 0 \qquad \text{for all } x \geq 2.$$

So, f is decreasing on $[2, \infty)$ and

$$\lim_{k\to\infty} \frac{1}{x\ln x} = 0$$

by the Alternating Series Test, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges.

$$\begin{aligned}
\int_2^{\infty} f(x)dx &= \int_2^{\infty} \frac{1}{x\ln x} dx \\
&= \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x} dx \\
&= \lim_{t\to\infty} [\ln|\ln x|]_2^t \\
&= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = +\infty
\end{aligned}$$

Thus, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges. We conclude that

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k} \text{ is conditionally convergent.}$$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k \sin k}{k!}$

Solution. Consider

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k \sin k}{k!} \right| = \sum_{k=1}^{\infty} \frac{|\sin k|}{k!}.$$

Since $0 \leq |\sin k| \leq 1$,

$$0 \leq \frac{|\sin k|}{k!} \leq \frac{1}{k!}.$$

Then

$$\lim_{k \to \infty} \left| \frac{1}{(k+1)!} \cdot k! \right| = \lim_{k \to \infty} \frac{1}{k+1} = 0 < 1.$$

By the Ratio Test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ converges. We conclude that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin k}{k!} \text{ is absolutely convergent.}$$

5. For each the following, find all values of $p \in \mathbb{R}$ for which the given series converges absolutely.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^p}{p^k}$

Solution. By The Ratio Test,

$$\lim_{k \to \infty} \left| \frac{(k+1)^p}{p^{k+1}} \cdot \frac{p^k}{k^p} \right| = \lim_{k \to \infty} \left| \frac{1}{p} \cdot \left(\frac{k+1}{k} \right)^p \right| = \left| \frac{1}{p} \right| < 1.$$

Then, $|p| > 1$. Therefore,

$$\sum_{k=1}^{\infty} \frac{k^p}{p^k} \text{ converges if and only if } |p| > 1.$$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{kp} k!}{k^k}$

Solution. By The Ratio Test,

$$\begin{aligned}
\lim_{k \to \infty} \left| \frac{2^{(k+1)p}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^{kp} k!} \right| &= \lim_{k \to \infty} \left| \frac{2^{kp+p}(k+1)!}{(k+1)^k (k+1)} \cdot \frac{k^k}{2^{kp} k!} \right| \\
&= \lim_{k \to \infty} 2^p \left(\frac{k}{k+1} \right)^k \\
&= 2^p \lim_{k \to \infty} \left(\frac{k}{k+1} \right)^k
\end{aligned}$$

Then,

$$\begin{aligned}
L &= \left(\frac{k}{k+1}\right)^k \\
\ln L &= k \ln\left(\frac{k}{k+1}\right) \\
\lim_{k\to\infty} \ln L &= \lim_{k\to\infty} k \ln\left(\frac{k}{k+1}\right) \\
&= \lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+1}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{k+1}{k}\cdot\left(\frac{k}{k+1}\right)'}{-\frac{1}{k^2}} \\
&= \lim_{k\to\infty} \frac{\frac{k+1}{k}\cdot\frac{1}{(k+1)^2}}{-\frac{1}{k^2}} \\
&= \lim_{k\to\infty} \frac{-k^2}{k(k+1)} = -1 \\
\lim_{n\to\infty} L &= e^{-1} = \frac{1}{e}
\end{aligned}$$

So,

$$\lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e}$$

We obtain

$$\lim_{k\to\infty} \left| \frac{2^{(k+1)p}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^{kp}k!} \right| = 2^p \cdot \frac{1}{e} < 1.$$

Then, $2^p < e$, i.e., $p < \log_2 e$. Therefore,

$$\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k} \text{ converges if and only if } p < \log_2 e.$$

6. Assume that $\sum_{k=1}^{\infty} a_k$ coverges absolutely. Use **Cauchy Criterion** to prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{k} \qquad \text{coverges absolutely.}$$

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ coverges absoluetly. Then $\sum_{k=1}^{\infty} |a_k|$ converges.
Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Let $m, n \in \mathbb{N}$ such that $m > n \geq N$. If $n \leq k \leq m$, then $\frac{1}{k} \leq 1$. We obatin

$$\left| \sum_{k=n}^{m} \frac{a_k}{k} \right| \leq \sum_{k=n}^{m} \left| \frac{a_k}{k} \right| = \sum_{k=n}^{m} \frac{|a_k|}{k} \leq \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Thus, $\sum_{k=1}^{\infty} |a_k|$ converges absolutely.

$\square$

7. Prove that

$$\sum_{k=1}^{\infty}(-1)^k \arctan\left(\frac{1}{k}\right)$$

is conditionally convergent.

Hint: Use Alternating Series Test and Limit Comparision Test.

Solution. Firstly, we see that

$$\lim_{k\to\infty} \arctan\left(\frac{1}{k}\right) = 0.$$

Next, let $f(x) = \arctan\left(\frac{1}{x}\right)$ where $x \geq 1$. The derivative of $f(x)$ is

$$f'(x) = \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{1+x^2} < 0 \quad \text{for all } x \geq 1.$$

So, $\left\{\arctan\left(\frac{1}{k}\right)\right\}$ is decreasing. By Alternating Series Test (AST),

$$\sum_{k=1}^{\infty}(-1)^k \arctan\left(\frac{1}{k}\right) \qquad \text{converges.}$$

Finally, we consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \arctan\left(\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty} \arctan\left(\frac{1}{k}\right)$$

and

$$\lim_{k\to\infty} \frac{\arctan\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{1}{1+\frac{1}{k^2}}\left(-\frac{1}{k^2}\right)}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^2}} = 1 > 0$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k}\right) \qquad \text{diverges.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \arctan\left(\frac{1}{k}\right) \qquad \text{is conditionally convergent.}$$