Solution Assignment 1
MAC3309 Mathematical Analysis

Topic	Ordered field axiom & Well-Ordering Principle	**Score**	10 marks
Time	$1st$ Week		
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1. Define $(\sqrt{a})^2 = a$ for all $a \geq 0$. Prove that

$$\sqrt{x^2} = |x| \quad \text{for all } x \in \mathbb{R}.$$

Proof. Let $x \in \mathbb{R}$. By the fact that $(\sqrt{x^2})^2 = x^2$ and $|x|^2 = x^2$,

$$(\sqrt{x^2})^2 - |x|^2 = x^2 - x^2 = 0$$
$$(\sqrt{x^2} - |x|)(\sqrt{x^2} + |x|) = 0.$$

Case $x = 0$. Then $\sqrt{x^2} = 0 = |x|$
Case $x \neq 0$. Then $\sqrt{x^2} > 0$ and $|x| > 0$. So, $\sqrt{x^2} + |x| > 0$. Then $(\sqrt{x^2} + |x|)^{-1} \in \mathbb{R}$. We obtain

$$\sqrt{x^2} - |x| = (\sqrt{x^2} - |x|)(\sqrt{x^2} + |x|)(\sqrt{x^2} + |x|)^{-1} = 0 \cdot (\sqrt{x^2} + |x|)^{-1} = 0$$

Thus, $\sqrt{x^2} = |x|$. The proof is complete. □

2. Let a and b be real numbers. Prove that

$$\text{if} \quad 0 < a < b, \quad \text{then} \quad \sqrt{a} < \sqrt{b}.$$

Proof. Let a and b be real numbers. Assume that $0 < a < b$. Then

$$a - b < 0$$
$$(\sqrt{a})^2 - (\sqrt{b})^2 < 0$$
$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) < 0$$

Since $\sqrt{a} + \sqrt{b}$ is always a positive number, $(\sqrt{a} + \sqrt{b})^{-1} > 0$. By O4.1, we obtain

$$\sqrt{a} - \sqrt{b} = (\sqrt{a} + \sqrt{b})^{-1}(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) < (\sqrt{a} + \sqrt{b})^{-1} \cdot 0 = 0.$$

Hence, $\sqrt{a} < \sqrt{b}$. □

3. Let $x \in \mathbb{R}$. Prove that

$$-1 \leq x \leq 2 \quad \text{implies} \quad |x^2 + x - 2| \leq 4|x - 1|.$$

Proof. Let $x \in \mathbb{R}$. Suppose that $-1 \leq x \leq 2$. Then

$$1 \leq x + 2 \leq 4.$$

It implies $|x + 2| \leq 4$. If $x = 1$, then $|x^2 + x - 2| = 0 \leq 0 = 4|x - 1|$.
For case $x \neq 1$, we give $|x - 1| > 0$. Thus,

$$|x + 2||x - 1| \leq 4|x - 1|$$
$$|x^2 + x - 2| \leq 4|x - 1|$$

□

4. Let x and y be two distinct real numbers. Prove that

$$\frac{x+y}{2} \text{ lies between } x \text{ and } y.$$

Proof. Let x and y be two distinct real numbers.
By Trinochomy rule, $x \neq y$. WLOG $x < y$. Then $x + x < x + y$ and $x + y < y + y$.
By transitive rule,

$$2x < x + y < 2y$$
$$x < \frac{x+y}{2} < y.$$

□

5. Let a and b be positive real numbers. Prove that

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proof. Let a and b be positive real numbers. Then $\sqrt{a}$ and $\sqrt{b}$ are reals.
By fact that $(\sqrt{a} - \sqrt{b})^2 \geq 0$, we give

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$
$$a + b \geq 2\sqrt{ab}$$

The proof is complete. □

6. Let a and b be positive real numbers. Use 5 to prove that

$$\frac{2ab}{a+b} \leq \sqrt{ab}.$$

Proof. Let a and b be positive real numbers. By 5., we give

$$\frac{1}{a} + \frac{1}{b} \geq 2\sqrt{\frac{1}{a} \cdot \frac{1}{b}}$$
$$\frac{a+b}{ab} \geq \frac{2}{\sqrt{ab}}$$

Since $\dfrac{ab\sqrt{ab}}{a+b}$ is posotive, by O4.1, we obtain

$$\frac{a+b}{ab} \cdot \frac{ab\sqrt{ab}}{a+b} \geq \frac{2}{\sqrt{ab}} \cdot \frac{ab\sqrt{ab}}{a+b}$$
$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

The proof is complete. □

7. Let $a, b, x, y \in \mathbb{R}$. Prove that

$$(ab + xy)^2 \leq (a^2 + x^2)(b^2 + y^2).$$

Proof. Let $a, b, x, y \in \mathbb{R}$. By the fact that $(ay - xb)^2 \geq 0$, we have

$$\begin{aligned}
(ab + xy)^2 &\leq (ab + xy)^2 + (ay - xb)^2 \\
&= (a^2b^2 + 2abxy + x^2y^2) + (a^2y^2 - 2abxy + x^2b^2) \\
&= a^2b^2 + a^2y^2 + x^2b^2 + x^2y^2 \\
&= a^2(b^2 + y^2) + x^2(b^2 + y^2) \\
&= (a^2 + x^2)(b^2 + y^2)
\end{aligned}$$

The proof is complete. □

8. Let a and b be real number. Use Triangle Inequality to prove that

$$||a| - |b|| \leq |a - b|$$

Proof. Let $a, b \in \mathbb{R}$. From $|a| - |b| \leq |a - b|$, we obtain

$$|a| \leq |a - b| + |b| \qquad (*)$$

From $|a| - |b| \leq |a + b|$, we substitue a and b by b and $a - b$, respectively. Then

$$|b| - |a - b| \leq |a| \qquad (**)$$

From $(*)$ and $(**)$, and by transitive law, we get

$$\begin{aligned} |b| - |a - b| \leq |a| \leq |a - b| + |b| \\ -|a - b| \leq |a| - |b| \leq |a - b| \end{aligned}$$

Thus, $||a| - |b|| \leq |a - b|$. □

9. Let $x, y \in \mathbb{R}$. Prove that

$$x > y - \varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{if and only if} \quad x \geq y$$

Proof. Let $x, y \in \mathbb{R}$.
$(\rightarrow)$ We will prove by contradiction. Assume that

$$x > y - \varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{and } x < y.$$

Then $y - x > 0$, it is a role of ε in the assumption. We obtain

$$\begin{aligned} x &> y - (y - x) \\ x &> x. \end{aligned}$$

This is contradiction because x is not greater than x.
$(\leftarrow)$ We will prove by contrapositive. Assume that there is $\varepsilon > 0$ such that $x \leq y - \varepsilon$. Then $-\varepsilon < 0$. We obtain $y - \varepsilon < y$. By assumption,

$$x \leq y - \varepsilon < y.$$

Thus, $x < y$. □

10. Prove **Mathematical Induction (Theorem 1.2.2 page19)**.

Mathematical Induction: Suppose for each $n \in \mathbb{N}$ that $P(n)$ is a statement that satisfies the following two properties:

(1) Basic step : $P(1)$ is true
(2) Inductive step : For every $k \in \mathbb{N}$ for which $P(k)$ is true, $P(k + 1)$ is also true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. We will prove by contradiction. Assume that (1) and (2) are ture and there is an $n_0 \in \mathbb{N}$ such that $P(n_0)$ is false. Define

$$S = \{n \in \mathbb{N} : P(n) \text{ is false } \}.$$

Then, $n_0 \in S \subseteq \mathbb{N}$. By WOP, S has a least element, said $m \in S$.
Since (1) is true, $m \neq 1$. Then $m > 1$ or $m - 1 > 0$. So, $m - 1 \in \mathbb{N}$.
But $m - 1 < m$ and m is the least element in S, so $m - 1 \notin S$. Set

$$k = m - 1 \in \mathbb{N}. \text{ We obtain } P(k) \text{ is true.}$$

By (2), $P(k + 1) = P(m)$ is true. This contradicts $m \in S$. □