Solution Assignment 2
MAC3309 Mathematical Analysis

Topic Completeness Axiom & Functions **Score** 10 marks
Time $2nd$ Week
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1. Let $A = \left\{\frac{n-1}{n+1} : n \in \mathbb{N}\right\}$. Find $\inf A$ and $\sup A$ with proving them.

   Claim that $\inf A = 0$ and $\sup A = 1$.

   Proof. $\inf A = 0$.
   Let $n \in \mathbb{N}$. Then $1 \le n$ or $0 \le n-1 < n+1$. It's clear that

   $$0 \le \frac{n-1}{n+1}.$$

   Thus, 0 is a lower bound of A.
   Let ℓ be a lower bound of A. For $n = 1$, we get $0 \in A$. So, $\ell \le 0$. Hence, $\inf A = 0$.

   $\sup A = 1$.
   Let $n \in \mathbb{N}$. Then $1 \le n$ or $0 \le n-1 \le n+1$. It's clear that

   $$\frac{n-1}{n+1} \le 1.$$

   Thus, 1 is an upper bound of A.
   Assume that that there is an upper bound u_0 of A such that

   $$u_0 < 1.$$

   By definition,

   $$\frac{n-1}{n+1} \le u_0 \quad \text{for all } \ n \in \mathbb{N} \qquad (*)$$

   Since $u_0 < 1$, $\frac{1-u_0}{2} > 0$. By Archimendean property, there is $n_0 \in \mathbb{N}$ such that

   $$\frac{1}{n_0} < \frac{1-u_0}{2}.$$

   By the fact that $n_0 + 1 > n_0$,

   $$\frac{1}{n_0+1} < \frac{1}{n_0} < \frac{1-u_0}{2}$$
   $$u_0 < 1 - \frac{2}{n_0+1} = \frac{n_0-1}{n_0+1}.$$

   This is contradiction to $(*)$. Therefore, $\sup A = 1$. □

2. Let $A = \left\{ \dfrac{1}{n^2+1} : n \in \mathbb{N} \right\}$. Find $\inf A$ and $\sup A$ with proving them.

   Claim that $\inf A = 0$ and $\sup A = \frac{1}{2}$.

   Proof. $\inf A = 0$.
   Let $n \in \mathbb{N}$. Then $n^2 + 1 \geq 0$. We obtain
   $$0 \leq \frac{1}{n^2+1}.$$
   Thus, 0 is a lower bound of A.
   Suppose that there is a lower bound ℓ_0 of A such that
   $$\ell_0 > 0.$$
   By definition,
   $$l_0 \leq \frac{1}{n^2+1} \quad \text{for all } \; n \in \mathbb{N}$$
   By Archimendean property, there is $n_0 \in \mathbb{N}$ such that
   $$\frac{1}{n_0} < \ell_0.$$
   Since $n_0^2 > n_0$ and $n_0^2 + 1 > n_0^2 > 0$,
   $$\frac{1}{n_0^2+1} < \frac{1}{n_0^2} < \frac{1}{n_0} < \ell_0.$$
   It contradicts to the lower bound ℓ_0. Hence, $\inf A = 0$.
   $\sup A = \frac{1}{2}$.
   Let $n \in \mathbb{N}$. Then $1 \leq n^2$. We get $2 < n^2 + 1$. So,
   $$\frac{1}{n^2+1} \leq \frac{1}{2}.$$
   Thus, $\frac{1}{2}$ is an upper bound of A.

   Let u be an upper bound of A. For $n = 1$, we get $\frac{1}{2} \in A$. So, $\frac{1}{2} \leq u$. Hence, $\inf A = \frac{1}{2}$. □

3. Prove **Approximation Property for Infimum (API)**
   If A has an infimum and $\varepsilon > 0$ is any positive number, then there is a point $a \in A$ such that
   $$\inf A \leq a < \inf A + \varepsilon$$

   Proof. Assume that A has an infimum, say ℓ_0. Suppose that there a positve $\varepsilon_0 > 0$ such that
   $$a < \ell_0 \quad \text{or} \quad a \geq \ell_0 + \varepsilon_0 \text{ for all } a \in A$$
   In this case $a < \ell_0$, it is imposible beacause ℓ_0 is a lower bound of A.
   From $a \geq \ell_0 + \varepsilon_0$ for all $a \in A$, it means that $\ell_0 + \varepsilon_0$ is a lower bound of A. But
   $$\ell_0 + \varepsilon_0 > \ell_0$$
   It's imposible because ℓ_0 is the greatest lower bound of A. □

4. Let r be a rational number and s be an irrational number. Prove that

4.1 $r+s$ is an irrational number.

Proof. Let r be a rational number and s be an irrational number. Then there are two integers p and q such that

$$r = \frac{p}{q} \text{ when } q \neq 0.$$

Suppose that $r+s$ is a rational number. Then there are two integers x and y such that

$$r + s = \frac{x}{y} \text{ when } y \neq 0.$$

So,

$$\begin{aligned} \frac{p}{q} + s &= \frac{x}{y} \\ s &= \frac{x}{y} - \frac{p}{q} = \frac{xq - py}{yq} \in \mathbb{Q} \end{aligned}$$

This is contradiction because $s \in \mathbb{Q}^c$. □

4.2 if $r \neq 0$, then rs is always an irrational number.

Proof. Let r be a non-zero rational number and s be an irrational number. Then there are two non-zero integers p and q such that

$$r = \frac{p}{q}$$

Suppose that rs is a rational number. Then there are two integers x and y such that

$$rs = \frac{x}{y} \text{ when } y \neq 0.$$

So,

$$\begin{aligned} \frac{p}{q} s &= \frac{x}{y} \\ s &= \frac{xq}{py} \in \mathbb{Q} \end{aligned}$$

This is contradiction because $s \in \mathbb{Q}^c$. □

5. Show that $\sqrt{2}$ is an irrational number.

Proof. Assume that $\sqrt{2}$ is a rational number. Then there are two integers p and q such that

$$\sqrt{2} = \frac{p}{q} \text{ when } q \neq 0 \text{ and } \gcd(p, q) = 1.$$

We have $2q^2 = p^2$. It implies that p is an even number. Then there is an $k \in \mathbb{Z}$ such that $p = 2k$. So,

$$\begin{aligned} 2q^2 &= (2k)^2 = 4k^2 \\ q^2 &= 2k^2 \end{aligned}$$

It implies again that q is an even number. Thus, $\gcd(p, q) \neq 1$. This is contradiction. □

6. Let $\sqrt{K} \in \mathbb{Q}^c$ and $a, b, x, y \in \mathbb{Z}$. Prove that

$$\text{if } a + b\sqrt{K} = x + y\sqrt{K}, \text{ then } a = x \text{ and } b = y.$$

Proof. Let $\sqrt{K} \in \mathbb{Q}^c$ and $a, b, x, y \in \mathbb{Z}$. Assume that $a + b\sqrt{K} = x + y\sqrt{K}$. Then

$$(a - x) + (b - y)\sqrt{K} = 0$$

Suppose that $b - y \neq 0$. Then

$$\sqrt{K} = -\frac{a - x}{b - y} \in \mathbb{Q}$$

This is contradiction to $\sqrt{K} \in \mathbb{Q}^c$. Thus, $b - y = 0$. It implies also that $a - x = 0$. $\square$

7. Prove **Theorem 1.3.13** : If x be a real number, then there exists an $n \in \mathbb{Z}$ such that

$$n - 1 \leq x < n.$$

Proof. Let $x \in \mathbb{R}$. If $x = 0$, we choose $n = 1$. We are done.
Case 1. $x > 0$. Define $S = \{n \in \mathbb{N} : n > x\} \subseteq \mathbb{N}$. By Archemedian property, $S \neq \varnothing$.
By WOP, S has the least element, say n_0. Since $n_0 - 1 < n_0$, $n_0 - 1 \notin A$. So, $n_0 - 1 \leq x$. Thus,

$$n_0 - 1 \leq x < n_0$$

The proof is complete in this case.
Case 2. $x < 0$. Then $-x > 0$. By Case 1, there is an $m \in \mathbb{N}$ such that $m - 1 \leq -x < m$. Then

$$-m < x \leq -m + 1$$

If $x = -m + 1$, we choose $n = -m + 2$. So,

$$n - 1 = -m + 1 = x < n \text{ or } n - 1 \leq x < n.$$

If $-m < x < -m + 1$, we choose $n = -m + 1$. So, $n - 1 < x < n$. It implies that

$$n - 1 \leq x < n.$$

$\square$

8. Use Theorem 1.3.13 to prove **Density of Rationals** :
If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is a rational number r such that

$$a < r < b.$$

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Then $b - a > 0$.
By AP, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{n} < b - a$. It follows that

$$na + 1 < nb.$$

By Theorem 1.3.13, there is an $m \in \mathbb{Z}$ such that $m - 1 \leq na < m$. It implies that

$$na < m \leq na + 1 < nb.$$

Set $r := \dfrac{m}{n}$. We obtain $a < r < b$. $\square$

9. Use the Density of Rationals to Prove **Density of Irratioals** :
If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is an irrational number t such that

$$a < t < b.$$

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. By the Density of Rational, there is an $r \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}.$$

It follows that

$$a < r\sqrt{2} < b.$$

If $r \neq 0$, then $t := r\sqrt{2}$ is irrational (see Exercise). It is done.
Case $r = 0$. By the Density of Rational, there is an $s \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < 0 < s < \frac{b}{\sqrt{2}}.$$

It follows that

$$a < s\sqrt{2} < b.$$

Set $t = s\sqrt{2}$, irrational. Thus, the proof is complete. □

10. Let $f(x) = x^2 e^{x^2}$ where $x \in \mathbb{R}$. Show that f is 1-1 on $(0, \infty)$.

Proof. Let $x_1, x_2 \in (0, \infty)$ such that $x_1 \neq x_2$. WLOG $x_1 > x_2 > 0$.
Then $x_1^2 > x_2^2 > 0$. We obtain $e^{x_1} > e^{x_2} > 0$. It implies that

$$x_1^2 e^{x_1} > x_2^2 e^{x_2}$$
$$f(x_1) > f(x_2).$$

Thus, $f(x_1) \neq f(x_2)$. We conclude that f is 1-1 on $(0, \infty)$. □