Math SSRU

Solution Assignment 3
MAC3309 Mathematical Analysis

Topic Limit of Sequences & Limit Theorems **Score** 10 marks
Time 3*rd* Week
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1. Use definition to prove that $\lim\limits_{n\to\infty} \dfrac{2n+1}{n+1}$ exists.

Proof. Let $\varepsilon > 0$. By Archimedean principle, there is $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\dfrac{1}{n} \leq \dfrac{1}{N}$. Since $n+1 > n$, $\dfrac{1}{n+1} < \dfrac{1}{n}$. Hence,

$$\left|\frac{2n+1}{n+1} - 2\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\lim\limits_{n\to\infty} \dfrac{2n+1}{n+1} = 2.$ □

2. Use definition to prove that $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1}$ exists.

Proof. Let $\varepsilon > 0$. Then $\sqrt{\varepsilon} > 0$. By Archimedean principle, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \sqrt{\varepsilon}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. Then $n^2 \geq N^2$. We obtain $\dfrac{1}{n^2} \leq \dfrac{1}{N^2}$. Since $n^2+1 > n^2$, $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$.
Hence,

$$\left|\frac{n^2}{n^2+1} - 1\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

Thus, $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1} = 1.$ □

3. Prove by contradiction to show that $\lim\limits_{n\to\infty} \sin\left(\dfrac{n\pi}{2}\right)$ does not exist (DNE).

Proof. Suppose that $\sin\left(\frac{n\pi}{2}\right) \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$.
Given $\varepsilon = 1$. There is an $N \in \mathbb{N}$, for all $n \geq \mathbb{N}$ implies

$$\left|\sin\left(\frac{n\pi}{2}\right) - a\right| < 1.$$

Since $\sin\left(\frac{n\pi}{2}\right)$ equals to either 0 or 1 or -1, we obtian $|0-a| < 1$ and $|1-a| < 1$ and $|-1-a| < 1$, i.e,

$$|a| < 1 \text{ and } |1-a| < 1 \text{ and } |1+a| < 1.$$

We have

$$2 = |(1+a) + (1-a)| \leq |1+a| + |1-a| < 1+1 = 2.$$

It is imposible. □

4. Assume that $x_n \to 1$ as $n \to \infty$. Show that

$$\frac{1}{(x_n)^2} \to 1 \text{ as } n \to \infty$$

Proof. Assume that $x_n \to 1$ as $n \to \infty$.
Given $\varepsilon = \frac{1}{2}$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - 1| < \tfrac{1}{2}.$$

Then

$$\begin{aligned} |x_n| - 1 \leq |x_n - 1| &\leq \frac{1}{2} \\ |x_n| &\leq \frac{3}{2} \end{aligned}$$

and

$$\begin{aligned} 1 = |1 - x_n + x_n| \leq |1 - x_n| + |x_n| &\leq \frac{1}{2} + |x_n| \\ \frac{1}{2} &\leq |x_n| \\ \frac{1}{|x_n|} &\leq 2 \end{aligned}$$

Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - 1| < \frac{\varepsilon}{10}.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\begin{aligned} \left| \frac{1}{(x_n)^2} - 1 \right| &= \left| \frac{1 - (x_n)^2}{(x_n)^2} \right| = \left| \frac{(1 - x_n)(1 + x_n)}{(x_n)^2} \right| \\ &\leq \frac{|1 - x_n||1 + x_n|}{|x_n|^2} \\ &\leq \frac{1}{|x_n|^2} \cdot (1 + |x_n|) \cdot |1 - x_n| \\ &< 2^2 \cdot \left(1 + \frac{3}{2}\right) \cdot \frac{\varepsilon}{10} = \varepsilon \end{aligned}$$

Thus, $\frac{1}{(x_n)^2} \to 1$ as $n \to \infty$ $\square$

5. Assume that $x_n \to 0$ as $n \to \infty$. Show that

$$\frac{1 + (x_n)^2}{x_n + 1} \to 1 \text{ as } n \to \infty$$

Proof. Assume that $x_n \to 0$ as $n \to \infty$.
Given $\varepsilon = \frac{1}{2}$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n| < \frac{1}{2}.$$

Then

$$\begin{aligned} 1 = |1 + x_n + x_n| \leq |1 + x_n| + |x_n| \leq |1 + x_n| + \frac{1}{2} \\ \frac{1}{2} \leq |1 + x_n| \\ \frac{1}{|1 + x_n|} \leq 2. \end{aligned}$$

Let $\varepsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n| < \frac{\varepsilon}{3}.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\begin{aligned} \left| \frac{1 + (x_n)^2}{x_n + 1} - 1 \right| &= \left| \frac{(x_n)^2 - x_n}{x_n + 1} \right| = \left| \frac{x_n(x_n - 1)}{x_n + 1} \right| \\ &\leq \frac{|x_n||x_n - 1|}{|x_n + 1|} = |x_n| \cdot \frac{1}{|x_n + 1|} \cdot (|x_n| + 1) \\ &\leq \frac{\varepsilon}{3} \cdot 2 \cdot (\frac{1}{2} + 1) = \varepsilon. \end{aligned}$$

Hence, Thus, $\frac{1 + (x_n)^2}{x_n + 1} \to 1$ as $n \to \infty$ □

6. Let $\alpha \in \mathbb{R}$ and $\{x_n\}$ be a convergent sequence. Prove that

$$\lim_{n \to \infty} (\alpha x_n) = \alpha \lim_{n \to \infty} x_n$$

Proof. Assume that $x_n \to a$ as $n \to \infty$.
Let $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. Then $|\alpha| + 1 > |\alpha| \geq 0$. So, $\frac{|\alpha|}{|\alpha| + 1} < 1$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{|\alpha| + 1}.$$

Let $n \in \mathbb{N}$. For each $n \geq N$, we obtain

$$|\alpha x_n - \alpha x| = |\alpha||x_n - x| < |\alpha| \cdot \frac{\varepsilon}{|\alpha| + 1} = \frac{|\alpha|}{|\alpha| + 1}\varepsilon < 1 \cdot \varepsilon = \varepsilon.$$

Thus, $\lim_{n \to \infty} (\alpha x_n) = \alpha a = \alpha \lim_{n \to \infty} x_n$. □

7. If A has a finite infimum, then there is a sequence $x_n \in A$ such that

$$x_n \to \inf A \quad \text{as} \quad n \to \infty.$$

Proof. Suppose A has a finite infimum. By API, there is $x \in A$ such that

$$\inf A \leq x \leq \inf A + \varepsilon \qquad \text{for all } \varepsilon > 0.$$

We construct a sequence $\{x_n\}$ by

$$\begin{aligned}
\varepsilon_1 &= 1, & \exists x_1 \in A \text{ such that} \quad & \inf A \leq x_1 \leq \inf A + 1 \\
\varepsilon_2 &= \frac{1}{2}, & \exists x_2 \in A \text{ such that} \quad & \inf A \leq x_2 \leq \inf A + \frac{1}{2} \\
\varepsilon_3 &= \frac{1}{3}, & \exists x_3 \in A \text{ such that} \quad & \inf A \leq x_3 \leq \inf A + \frac{1}{3} \\
& & \vdots & \\
\varepsilon_n &= \frac{1}{n}, & \exists x_n \in A \text{ such that} \quad & \inf A \leq x_n \leq \inf A + \frac{1}{n}
\end{aligned}$$

Thus, $\{x_n\}$ is a sequence in A and satisfies

$$\inf A \leq x_n < \inf A + \frac{1}{n}$$

By the Squeez Theorem,

$$\begin{aligned}
\lim_{n\to\infty} \inf A \leq \lim_{n\to\infty} x_n &\leq \lim_{n\to\infty} \left(\inf A + \frac{1}{n} \right) \\
\inf A \leq \lim_{n\to\infty} x_n &\leq \inf A
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} x_n = \inf A.$$

□

8. If $\{x_n\}$ is a convergent sequence, then

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$$

when $\lim\limits_{n\to\infty} x_n \neq 0$ and $x_n \neq 0$.

Proof. Assume that $\{x_n\}$ converges to a such that $a \neq 0$. Let $\varepsilon > 0$. There is an $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \tfrac{|a|}{2}.$$

Then

$$\begin{aligned} |a| = |a - x_n + x_n| \leq |x_n - a| + |x_n| &\leq \frac{|a|}{2} + |x_n| \\ \frac{|a|}{2} &\leq |x_n| \\ \frac{1}{|x_n|} &\leq \frac{2}{|a|} \end{aligned}$$

There is an $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - a| < \frac{|a|^2}{2}\varepsilon.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ such that $n \geq N$. We have

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{a}\right| &= \left|\frac{a - x_n}{a x_n}\right| \\ &\leq \frac{1}{|x_n|} \cdot \frac{|x_n - a|}{|a|} \\ &< \frac{2}{|a|} \cdot \frac{|a|^2}{2|a|}\varepsilon = \varepsilon \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{\lim\limits_{n\to\infty} x_n}$. □

9. Let $\{x_n\}$ be convergent such that converges to a. Prove that

$$\lim_{n\to\infty} |x_n| = |a|.$$

Proof. Assume that $\{x_n\}$ converges to a. Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \varepsilon.$$

For each $n \in \mathbb{N}$ such that $n \geq N$, by part 4 of the Apply Triangle Inequality,

$$||x_n| - |a|| \leq |x_n - a| < \varepsilon.$$

Therefore, $|x_n| \to |a|$ as $n \to \infty$. □

10. Let $x_n > 0$ such that converges to $a > 0$, then prove that

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{a}.$$

Proof. Assume that $x_n > 0$ converges to $a > 0$. Then $\sqrt{a} > 0$.
Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |x_n - a| < \varepsilon\sqrt{a}.$$

Since $\sqrt{x_n} > 0$, $\sqrt{x_n} + \sqrt{a} > \sqrt{a}$. It follows that

$$\frac{1}{\sqrt{x_n} + \sqrt{a}} < \frac{1}{\sqrt{a}}.$$

For each $n \in \mathbb{N}$ such that $n \geq N$, we obtain

$$\begin{aligned} |\sqrt{x_n} - \sqrt{a}| &= \left|(\sqrt{x_n} - \sqrt{a}) \cdot \frac{\sqrt{x_n} + \sqrt{a}}{\sqrt{x_n} + \sqrt{a}}\right| \\ &= \left|\frac{x_n - a}{\sqrt{x_n} + \sqrt{a}}\right| = \frac{1}{\sqrt{x_n} + \sqrt{a}} \cdot |x_n - a| \\ &< \frac{1}{\sqrt{a}} \cdot |x_n - a| \\ &< \frac{1}{\sqrt{a}} \cdot \varepsilon\sqrt{a} = \varepsilon. \end{aligned}$$

Therefore, $\sqrt{x_n} \to \sqrt{a}$ as $n \to \infty$. □