Assignment 4
MAC3309 Mathematical Analysis

Topic Divergence, Monotone & Cauchy Sequences **Score** 10 marks

Time $4th$ Week

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1. Suppose that $x_n \to \infty$ as $n \to \infty$. Show that if $\{y_n\}$ is bounded and $x_n \neq 0$, then

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

2. Prove **the Comparison Theorem (Theorem 2.2.12)** :
Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \text{ for all } n \geq N_0, \quad \text{then } \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

3. Prove that

$$\lim_{n\to\infty} \frac{n^2}{1+2n} = +\infty.$$

4. Prove that

$$\lim_{n\to\infty} \frac{2-n^2}{2+n} = -\infty.$$

5. (**Theorem 2.2.20**) Let $\{x_n\}$ be a real sequence and $\alpha > 0$. Prove that

$$\text{if } x_n \to -\infty \text{ as } n \to \infty, \text{ then } \lim_{n\to\infty} (\alpha x_n) = -\infty.$$

6. (**Theorem 2.2.22**) Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that

$$y_n > K \text{ for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

Prove that if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$.

7. Prove that $\left\{\dfrac{1}{n^2}\right\}$ is Cauchy.

8. Prove that the sum of two Cauchy sequence is Cauchy.

9. Prove that any real sequence $\{x_n\}$ that satisfies

$$|x_n - x_{n+1}| \leq \frac{1}{2^{n+1}}, \qquad n \in \mathbb{N}$$

is convergent by showing the sequence is Cauchy. (Use the fact that $n < 2^n$ for all $n \in \mathbb{N}$)

10. Use the MCT to prove **Theorem 2.3.4** : if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.