Math SSRU

Solution Assignment 4
MAC3309 Mathematical Analysis

Topic Divergence, Monotone & Cauchy Sequences **Score** 10 marks
Time $4th$ Week
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1. Suppose that $x_n \to \infty$ as $n \to \infty$. Show that if $\{y_n\}$ is bounded and $x_n \neq 0$, then

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Proof. Suppose $x_n \to \infty$ as $n \to \infty$ and $\{y_n\}$ is bounded and $x_n \neq 0$.
Let $\varepsilon > 0$. Then there is a $K > 0$ such that $|y_n| \leq K$ for all $n \in \mathbb{N}$.
Set $M = \dfrac{K}{\varepsilon} > 0$. There is an $N \in \mathbb{N}$, $n \geq N$ implies $x_n > M$ or $x_n > \dfrac{K}{\varepsilon}$. Then $x_n > 0$ for $n \geq N$ and

$$\frac{1}{|x_n|} = \frac{1}{x_n} \leq \frac{\varepsilon}{K}.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\left|\frac{y_n}{x_n}\right| = |y_n| \cdot \frac{1}{|x_n|} < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Hence, $\displaystyle\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$ □

2. Prove **the Comparison Theorem (Theorem 2.2.12)** :
Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \text{ for all } n \geq N_0, \quad \text{then} \quad \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. Let $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Assume that there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \quad \text{for all } n \geq N_0.$$

Suppose that $\displaystyle\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$, i.e., $a > b$. Then $a - b > 0$. By assumption, there is an $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - a| < \frac{a-b}{2}$$
and
$$n \geq N_2 \quad \text{implies} \quad |y_n - b| < \frac{a-b}{2}.$$

For each $n \geq \max\{N_0, N_1, N_2\}$, it follows that

$$y_n < b + \frac{a-b}{2} = a - \frac{a-b}{2} < x_n$$

which contradics the assumption. Thus, $a \leq b$. □

3. Prove that

$$\lim_{n\to\infty} \frac{n^2}{1+2n} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{4M+1}{2} < N$. It is equivalent to

$$\frac{1}{4}(2n-1) > \frac{1}{4}(2N-1) > M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $2n-1 > 2N-1$. Since $0 > -\frac{1}{4}$, $n^2 > n^2 - \frac{1}{4}$. We obtain

$$\frac{n^2}{1+2n} > \frac{n^2-\frac{1}{4}}{1+2n} = \frac{\frac{1}{4}(2n-1)(2n+1)}{1+2n} = \frac{1}{4}(2n-1) > \frac{1}{4}(2N-1) > M.$$

Hence, $\lim\limits_{n\to\infty} \frac{n^2}{1+2n} = +\infty$. □

4. Prove that

$$\lim_{n\to\infty} \frac{2-n^2}{2+n} = -\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $2-M < N$. It is equivalent to

$$2-N < M.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then $-n \leq -N$. So, $2-n \leq 2-N$. We obtain

$$\frac{2-n^2}{2+n} = \frac{-n^2+2}{2+n} < \frac{-n^2+4}{2+n} = \frac{(2-n)(2+n)}{2+n} = 2-n < 2-N < M.$$

Hence, $\lim\limits_{n\to\infty} \frac{2-n^2}{2+n} = -\infty$. □

5. (**Theorem 2.2.20**) Let $\{x_n\}$ be a real sequence and $\alpha > 0$. Prove that

$$\text{if } x_n \to -\infty \text{ as } n \to \infty, \text{ then } \lim_{n\to\infty} (\alpha x_n) = -\infty.$$

Proof. Assume that $x_n \to -\infty$ as $n \to \infty$.
Let $M \in \mathbb{R}$ and $\alpha > 0$. By assumption, there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < \frac{M}{\alpha}.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\alpha x_n < \alpha \cdot \frac{M}{\alpha} = M.$$

Thus, $\lim\limits_{n\to\infty} \alpha x_n = -\infty$. □

6. (**Theorem 2.2.22**) Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that

$$y_n > K \text{ for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

Prove that if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that

$$y_n > K \quad \text{for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

Assume that $x_n \to -\infty$ as $n \to \infty$.
Let $M \in \mathbb{R}$.
Case $M = 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, we obtain

$$x_n \cdot y_n < 0 = M$$

Case $M > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < -\frac{M}{K} < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, $-y_n < -K < 0$. We obtain

$$x_n \cdot y_n < -\frac{M}{K} \cdot y_n = \frac{M}{K} \cdot (-y_n) < \frac{M}{K} \cdot (-K) = -M < 0 < M.$$

Case $M < 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad x_n < \frac{M}{K} < 0.$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Since $y_n > K > 0$, $-y_n < -K < 0$. We obtain

$$x_n \cdot y_n < \frac{M}{K} \cdot y_n = \frac{-M}{K} \cdot (-y_n) < \frac{-M}{K} \cdot (-K) = M.$$

Thus, $\lim_{n\to\infty} x_n y_n = -\infty$. □

7. Prove that $\left\{\frac{1}{n^2}\right\}$ is Cauchy.

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\frac{1}{m} \leq \frac{1}{N}$ and $\frac{1}{n} \leq \frac{1}{N}$. Since $n^2 \geq n$ and $m^2 \geq m$, $\frac{1}{m^2} \leq \frac{1}{m}$ and $\frac{1}{n^2} \leq \frac{1}{n}$. We obtain

$$\left|\frac{1}{m^2} - \frac{1}{n^2}\right| \leq \frac{1}{m^2} + \frac{1}{n^2} \leq \frac{1}{m} + \frac{1}{n} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon.$$

Thus, $\left\{\frac{1}{n^2}\right\}$ is Cauchy. □

8. Prove that the sum of two Cauchy sequence is Cauchy.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy. Let $\varepsilon > 0$. There are $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} m, n \geq N_1 &\quad \rightarrow \quad |x_n - x_m| < \frac{\varepsilon}{2} \\ m, n \geq N_2 &\quad \rightarrow \quad |y_n - y_m| < \frac{\varepsilon}{2} \end{aligned}$$

Choose $N = \max\{N_1, N_2\}$. For $m, n \geq N_1$, we obtain

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &\leq |x_n - x_m| + |y_n - y_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, $\{x_n + y_n\}$ is Cauchy. □

9. Prove that any real sequence $\{x_n\}$ that satisfies

$$|x_n - x_{n+1}| \leq \frac{1}{2^{n+1}}, \qquad n \in \mathbb{N}$$

is convergent by showing the sequence is Cauchy. (Use the fact that $n < 2^n$ for all $n \in \mathbb{N}$)

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$. By the fact that $n < 2^n$ for all $n \in \mathbb{N}$, we get $\frac{1}{2^n} < \frac{1}{n}$.
Suppose that $m > n$. We obtain

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} - \cdots + x_{m-1} - x_m| \\ &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \\ &= \frac{1}{2^{n-1}} \left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}} \right] \\ &= \frac{1}{2^{n-1}} \sum_{k=1}^{m-n} \frac{1}{2^k} \\ &= \frac{1}{2^n} \left[1 - \frac{1}{2^{m-n}} \right] \\ &\leq \frac{1}{2^n} && \because 1 - \frac{1}{2^{m-n}} \leq 1 \quad \text{when } m - n > 0 \\ &< \frac{1}{n} && \because n < 2^n \\ &< \frac{1}{N} < \varepsilon \end{aligned}$$

Thus, $\{x_n\}$ is Cauchy. Therefore, $\{x_n\}$ is convergent. □

10. Use the MCT to prove **Theorem 2.3.4** : if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.

Proof. Assume that $|a| < 1$.
Case 1 $a = 0$. Then $a^n = 0$ for all $n \in \mathbb{N}$, and it follows that $a^n \to 0$ as $n \to \infty$.
Case 2 $a \neq 0$. Then $|a| > 0$. So, $0 < |a| < 1$. We obtain

$$0 < |a|^{n+1} < |a|^n < 1 \qquad \text{for all } n \in \mathbb{N}.$$

So, $\{|a|^n\}$ is decreasing and bounded below by 1 . By MCT, $|a|^n \to L$ as $n \to \infty$.
Suppose that $L \neq 0$. Then

$$L = \lim_{n\to\infty} |a|^{n+1} = \lim_{n\to\infty} |a|^n |a| = |a| \lim_{n\to\infty} |a|^n = |a| L.$$

We have $|a| = 1$ which contradics $|a| < 1$. Thus, $L = 0$. □