Solution Assignment 5
MAC3309 Mathematical Analysis

Topic	Topology on $\mathbb{R}$ and Limit of functions	**Score**	10 marks
Time	$5th$ Week		
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1. Let $U \subseteq \mathbb{R}$ be a nonempty open set. Assume that U has a supremum and infimum. Show that

$$\sup U \notin U \text{ and } \inf U \notin U.$$

Proof. Let $U \subseteq \mathbb{R}$ be a nonempty open set. Assume that U has a supremum and infimum. Suppose that $s := \sup U \in U$. Since U is open, there is $\varepsilon > 0$ such that

$$(s-\varepsilon, s) \cup (s, s+\varepsilon) \subseteq U.$$

Then $s < \frac{s+(s+\varepsilon)}{2} < s+\varepsilon$, i.e., $\frac{s+(s+\varepsilon)}{2} \in U$. This is contraction to s is an upper bound of U. Suppose that $\ell := \inf U \in U$. Since U is open, there is $\zeta > 0$ such that

$$(\ell-\zeta, \ell) \cup (\ell, \ell+\zeta) \subseteq U.$$

Then $\ell - \zeta < \frac{(\ell-\zeta)+\ell}{2} < \ell$, i.e., $\frac{(\ell-\zeta)+\ell}{2} \in U$. This is contraction to ℓ is a lower bound of U. □

2. Prove **Theorem 3.3.11** : Let $A \subseteq \mathbb{R}$. Then $\bar{A}$ is closed.

Proof. Let $x \in (\bar{A})^c = (A \cup A')^c$. Then $x \notin A$ and $x \notin A'$. There is an $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \cap A = [(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A = \varnothing.$$

Since $x \notin A$, $(x-\varepsilon, x+\varepsilon) \cap A = [(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A$. Use the fact that $A \subseteq \bar{A}$, we obtain

$$(x-\varepsilon, x+\varepsilon) \cap \bar{A} = \varnothing.$$

So, $(x-\varepsilon, x+\varepsilon) \subseteq (\bar{A})^c$. Thus, $(\bar{A})^c$ is open. We conclude that $\bar{A}$ is closed. □

3. Let A and B be subsets of $\mathbb{R}$. Prove that

$$(A \cup B)' = A' \cup B'.$$

Use the resulte to confirm that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. Let A and B be subsets of $\mathbb{R}$.
Let $x \in (A \cup B)'$. Then, for all $\varepsilon > 0$, we obtain

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap (A \cup B) \neq \varnothing$$
$$[[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A] \cup [[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap B] \neq \varnothing$$

Then,

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A \neq \varnothing \text{ or } [(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap B \neq \varnothing.$$

So, $x \in A' \cup B' \subseteq \bar{A} \cup \bar{B}$. Thus, $(A \cup B)' \subseteq A' \cup B'$.
Let $x \in A' \cup B'$. WLOG, let $x \in A'$. Then, for all $\varepsilon > 0$, we obtain

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A \neq \varnothing.$$

Since $A \subseteq A \cup B$,

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap (A \cup B) \neq \varnothing.$$

So, $x \in (A \cup B)'$. Thus, $A' \cup B' \subseteq (A \cup B)'$. We conclude that $(A \cup B)' = A' \cup B'$.
This result will confirm that

$$\begin{aligned} \overline{A \cup B} &= (A \cup B) \cup (A \cup B)' \\ &= (A \cup B) \cup (A' \cup B') \\ &= (A \cup A') \cup (B \cup B') \\ &= \bar{A} \cup \bar{B} \end{aligned}$$

Thus, $\overline{A \cup B} = \bar{A} \cup \bar{B}$. □

4. Prove converse of **Theorem 3.3.13** :
If the limit of every convergent sequence in F belongs to $F \subseteq \mathbb{R}$, then F is closed.

Proof. We will prove by contrapositive.
Assume that F is not closed. Then F^c is not open. There is an $x \in F^c$ such that

$$(x-\delta, x+\delta) \nsubseteq F^c \quad \text{for all } \delta > 0.$$

It follows that

$$(x-\delta, x+\delta) \cap F \neq \varnothing \quad \text{for all } \delta > 0.$$

Choose $\delta = \dfrac{1}{n}$ and $x_n \in \left(x - \dfrac{1}{n}, x + \dfrac{1}{n}\right) \cap F$ for each $n \in \mathbb{N}$. Then x_n is a sequence in F. It implies that

$$|x_n - x| < \frac{1}{n}.$$

We obtain that $x_n \to x$ as $n \to \infty$.
We conclude that there is a convergent sequence in F such that the limit is not in F. □

5. Use definition to prove that $\lim\limits_{x \to 1} x^2 + x + 1 = 3$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \dfrac{\varepsilon}{4}\right\}$. Suppose that $0 < |x-1| < \delta$. Then $0 < |x-1| < 1$

$$\begin{aligned} |x| - 1 \leq |x-1| &< 1 \\ |x| &< 2 \end{aligned}$$

We obtain

$$\begin{aligned} |(x^2+x+1) - 3| &= |x^2 + x - 2| \\ &= |(x+2)(x-1)| \\ &= |x+2||x-1| \\ &< (|x|+2)\delta \\ &< (2+2)\frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x \to 1} x^2 + x + 1 = 3$. □

6. Use definition to prove that $\lim_{x\to -1} x^2 - x + 1 = 3$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{4}\right\}$. Suppose that $0 < |x+1| < \delta$. Then $0 < |x+1| < 1$

$$\begin{aligned} |x| - 1 \le |x+1| &< 1 \\ |x| &< 2 \end{aligned}$$

We obtain

$$\begin{aligned} |(x^2 - x + 1) - 3| &= |x^2 - x - 2| \\ &= |(x-2)(x+1)| \\ &= |x-2||x+1| \\ &< (|x| + 2)\delta \\ &< (2+2)\frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, $\lim_{x\to -1} x^2 - x + 1 = 3$. □

7. Use definition to prove that $\lim_{x\to 0} \frac{x^2+1}{x+1} = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{0.5, \frac{\varepsilon}{3}\right\}$. Suppose that $0 < |x| < \delta$. Then $0 < |x| < 0.5$,

$$-0.5 < x < 0 \quad \text{or} \quad 0 < x < 0.5. \text{ So, } 0.5 < x + 1 < 1 \quad \text{or} \quad 1 < x + 1 < 1.5.$$

Thus, $0.5 < |x+1| < 1.5$. We get $\frac{1}{|x+1|} < 2$. Then,

$$\begin{aligned} \left|\frac{x^2+1}{x+1} - 1\right| &= \left|\frac{x^2 - x}{x+1}\right| \\ &= \left|\frac{x(x-1)}{x+1}\right| \\ &= |x| \cdot |x-1| \cdot \frac{1}{|x+1|} \\ &< \delta \cdot (|x| + 1) \cdot 2 \\ &< \frac{\varepsilon}{3} \cdot 1.5 \cdot 2 = \varepsilon. \end{aligned}$$

Therefore, $\lim_{x\to 0} \frac{x^2+1}{x+1} = 1$. □

8. Use definition to prove that $\lim_{x\to 0} \frac{x^2+1}{x-1} = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{0.5, \frac{\varepsilon}{3}\right\}$. Suppose that $0 < |x| < \delta$. Then $0 < |x| < 0.5$,

$$-0.5 < x < 0 \quad \text{or} \quad 0 < x < 0.5. \text{ So, } -1.5 < x - 1 < -1 \quad \text{or} \quad -1 < x - 1 < -0.5.$$

Thus, $0.5 < |x-1| < 1.5$. We get $\dfrac{1}{|x-1|} < 2$. Then,

$$\begin{aligned}
\left|\frac{x^2+1}{x-1}+1\right| &= \left|\frac{x^2+x}{x+1}\right| \\
&= \left|\frac{x(x+1)}{x-1}\right| \\
&= |x|\cdot|x+1|\cdot\frac{1}{|x-1|} \\
&< \delta\cdot(|x|+1)\cdot 1 \\
&< \frac{\varepsilon}{3}\cdot 1.5\cdot 2 = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x\to 0}\dfrac{x^2+1}{x-1} = -1$. □

9. Let $y = f(x)$ be a real value function. Assume that

$$\lim_{x\to 1}\frac{f(x)}{x-1} \quad \text{exists.}$$

Prove that $\lim_{x\to 1} f(x) = 0$.

Proof. Assume that $\dfrac{f(x)}{x-1} \to a$ as $x \to 1$ for some $a \in \mathbb{R}$.
Given $\varepsilon = 1$. Then there $\delta_0 > 0$ such that $0 < |x-1| < \delta_0$, implies that

$$\left|\frac{f(x)}{x-1} - a\right| < 1.$$

Let $\varepsilon > 0$. Choose $\delta = \min\{\frac{\varepsilon}{1+|a|}, \delta_0\}$. Then $0 < |x-1| < \delta$. We obtain

$$\begin{aligned}
|f(x) - 0| = |f(x)| &= \left|\frac{f(x)}{x-1}\cdot(x-1)\right| \\
&= \left|\frac{f(x)}{x-1}\right| |x-1| \\
&= \left|\frac{f(x)}{x-1} - a + a\right| |x-1| \\
&\le \left(\left|\frac{f(x)}{x-1} - a\right| + |a|\right) |x-1| \\
&< (1+|a|)|x-1| \\
&< (1+|a|)\cdot\frac{\varepsilon}{1+|a|} = \varepsilon.
\end{aligned}$$

□

10. Let $y = f(x)$ be a real value function. Assume that

$$\lim_{x\to 0} \frac{f(x)}{x} \quad \text{exists.}$$

Prove that $\lim_{x\to 0} f(x) = 0$.

Proof. Assume that $\frac{f(x)}{x} \to a$ as $x \to 0$ for some $a \in \mathbb{R}$.
Given $\varepsilon = 1$. Then there $\delta_0 > 0$ such that $0 < |x| < \delta_0$, it implies that

$$\left|\frac{f(x)}{x} - a\right| < 1.$$

Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{\varepsilon}{1+|a|}, \delta_0\right\}$. Then $0 < |x| < \delta$. We obtain

$$\begin{aligned}
|f(x) - 0| = |f(x)| &= \left|\frac{f(x)}{x} \cdot x\right| \\
&= \left|\frac{f(x)}{x}\right| |x| \\
&= \left|\frac{f(x)}{x} - a + a\right| |x| \\
&\leq \left(\left|\frac{f(x)}{x} - a\right| + |a|\right) |x| \\
&< (1 + |a|)\, |x| \\
&< (1 + |a|) \cdot \frac{\varepsilon}{1+|a|} = \varepsilon.
\end{aligned}$$

□