Math SSRU

Solution Assignment 6
MAC3309 Mathematical Analysis

Topic	Limit Theorems of functions, One-sided Limits and Infinite Limits	**Score**	10 marks
Time	$6th$ Week		
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University		

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1. Let $f(x) = \begin{cases} 2x+1 & \text{if } x > 1 \\ x+2 & \text{if } x \le 1 \end{cases}$. Use definition to prove that $\lim_{x\to 1} f(x)$ exists.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{\varepsilon}{2}, \varepsilon\right\}$. Let $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$.
Case $0 < x-1 < \delta$. Then $x > 1$. We obtain

$$|f(x) - 3| = |2x+1-3| = |2x-2| = 2|x-1| < 2\delta < \varepsilon.$$

Case $-\delta < x-1 < 0$. Then $x < 1$. We obtain

$$|f(x) - 3| = |x+2-3| = |x-1| < \delta < \varepsilon.$$

We conclude that $\lim_{x\to 1} f(x) = 3$. □

2. Use definition to prove that $\lim_{x\to\infty} \frac{x}{x-1} = 1$

Proof. Let $\varepsilon > 0$. Choose $M = 1 + \frac{1}{\varepsilon}$. Then $M - 1 = \frac{1}{\varepsilon} > 0$ i.e., $M > 1$.
Let $x \in \mathbb{R}$ such that $x > M > 1$. It follows that $x - 1 > M - 1 > 0$. So, $\frac{1}{x-1} < \frac{1}{M-1}$.
We obtain

$$\left|\frac{x}{x-1} - 1\right| = \left|\frac{1}{x-1}\right| = \frac{1}{x-1} < \frac{1}{M-1} = \varepsilon$$

Thus, $\lim_{x\to\infty} \frac{x}{x-1} = 1$. □

3. Use definition to prove that $\lim_{x\to-\infty} \frac{x}{x-1} = 1$

Proof. Let $\varepsilon > 0$. Choose $M = -\frac{1}{\varepsilon}$. Then $M < 0$.
It's clear that $0 < -M < -M + 1$. So, $\frac{1}{1-M} < \frac{1}{-M}$.
Let $x \in \mathbb{R}$ such that $x < M$. It follows that $-x > -M > 0$, i.e., $1 - x > 1 - M > 1$ So, $\frac{1}{1-x} < \frac{1}{1-M}$.
We obtain

$$\left|\frac{x}{x-1} - 1\right| = \left|\frac{1}{x-1}\right| = \frac{1}{1-x} < \frac{1}{1-M} < \frac{1}{-M} = \varepsilon$$

Thus, $\lim_{x\to-\infty} \frac{x}{x-1} = 1$. □

4. Use definition to prove that $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$

Proof. Let $M > 0$. Choose $\delta = \frac{1}{M}$. Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < x - 1 < \delta$. It follows that

$$\frac{1}{x-1} > \frac{1}{\delta} = M$$

Thus, $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$. □

5. Use definition to prove that

$$\lim_{x\to 1^-} \frac{1}{1-x} = +\infty.$$

Proof. Let $M > 0$. Choose $\delta = \frac{1}{M}$. Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $-\delta < x - 1 < 0$. Then $0 < 1 - x < \delta$. It follows that

$$\frac{1}{1-x} > \frac{1}{\delta} = M$$

Thus, $\lim_{x\to 1^-} \frac{1}{1-x} = +\infty$. □

6. Use definition to prove that $\lim_{x\to 2^+} \frac{1}{2-x} = -\infty$

Proof. Let $M < 0$. Choose $\delta = -\frac{1}{M}$. Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < x - 2 < \delta$. So, $-\delta < 2 - x < 0$. We obatin

$$\frac{1}{2-x} < -\frac{1}{\delta} = M$$

Thus, $\lim_{x\to 2^+} \frac{1}{2-x} = -\infty$. □

7. Use definition to prove that

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.$$

Proof. Let $M < 0$. Choose $\delta = -\frac{1}{M}$. Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $-\delta < x - 2 < 0$. We obatin

$$\frac{1}{x-2} < -\frac{1}{\delta} = M$$

Thus, $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$. □

8. Let f and g be real functions defined everywhere on I except possibly at a such that

$$f(x) \leq g(x) \quad \text{for all } x \in I - \{a\}.$$

Prove that if $f(x) \to \infty$ as $x \to a$, then $g(x) \to \infty$ as $x \to a$.

Proof. Assume that $f(x) \leq g(x)$ for all $x \in I - \{a\}$ and $f(x) \to \infty$ as $x \to a$.
Let $M > 0$. There is a $\delta_0 > 0$ such that

$$\text{for all } x \in \mathbb{R} \text{ satisfying } 0 < |x - a| < \delta_0, \text{ implies that } f(x) > M.$$

Since $f(x) \leq g(x)$ for $x \in \mathbb{R}$ satisfying $0 < |x - a| < \delta_0$,

$$g(x) \geq f(x) > M.$$

Therefore, $g(x) \to \infty$ as $x \to a$. $\square$