Solution Assignment 7
MAC3309 Mathematical Analysis

Topic Continuity & Uniform continuity **Score** 10 marks
Time $9th$ Week
Teacher Assistant Professor Thanatyod Jampawai, Ph.D.
Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

---

1. Use definition to prove that $f(x) = \dfrac{1}{x}$ is continuous at $x = 1$.

   Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{0.5, \frac{\varepsilon}{2}\}$ such that $|x-1| < \delta$. Then $|x-1| < 0.5$. So,

   $$-0.5 < x - 1 < 0.5 \text{ or } 0.5 < |x| < 1.5.$$

   Thus, $\dfrac{1}{|x|} < 2$. We obtain

   $$\begin{aligned} |f(x) - f(1)| &= \left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| \\ &= \frac{1}{|x|} \cdot |x-1| < 2\delta < 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

   Therefore, f is continuous at $x = 1$. □

2. Prove that if f is continuous at a, then

   $$\lim_{h \to 0} f(a+h) = f(a).$$

   Proof. Assume that f is continuous at a. Let $\varepsilon > 0$. There is a $\delta > 0$ such that

   $$|x - a| < \delta \quad \text{imples} \quad |f(x) - f(a)| < \varepsilon \qquad (*)$$

   Let $h \in \mathbb{R}$ such that $0 < |h| < \delta$. Choose $x = a + h$. Then $0 < |x - a| = |h| < \delta$ satisfying $(*)$. So,

   $$|f(a+h) - f(a)| < \varepsilon.$$

   Hence, $f(a+h) \to f(a)$ as $h \to 0$.

   □

3. Prove that if $\lim\limits_{h \to 0} f(a+h) = f(a)$, then

   $$f \text{ is continuous at } a.$$

   Proof. Assume that $\lim\limits_{h \to 0} f(a+h) = f(a)$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that

   $$0 < |h| < \delta \quad \text{imples} \quad |f(a+h) - f(a)| < \varepsilon \qquad (**)$$

Let $x \in \mathbb{R}$ such that $|x-a| < \delta$.
Case $x \neq a$. Choose $h = x - a$. Then $0 < |h| = |x-a| < \delta$. By (**), $|f(a+(x-a)) - f(a)| < \varepsilon$, i.e.,

$$|f(x) - f(a)| < \varepsilon.$$

$$|f(x) - f(a)| = |f(a+h) - f(a)| < \varepsilon$$

Case $x = a$. It's clear that $|f(x) - f(a)| = 0 < \varepsilon$.
Therefore, f is continuous at a. □

4. Let E be a nonempty subset of $\mathbb{R}$ and $a \in E$. Suppose that $f : E \to \mathbb{R}$ is continuous at $a \in E$. Prove that

$$\text{If } x_n \text{ converges to } a \text{ and } x_n \in E, \text{ then } f(x_n) \to f(a) \text{ as } n \to \infty.$$

Proof. Assume that f is continuous at $a \in E$ and x_n converges to a and $x_n \in E$.
Let $\varepsilon > 0$. There is an $\delta > 0$ such that

$$|x-a| < \delta \text{ implies that } |f(x) - f(a)| < \varepsilon. \qquad ... (*)$$

From x_n converges to a, there is an $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies that } |x_n - a| < \delta. \qquad ... (**)$$

Let $n \in \mathbb{N}$ such that $n \geq N$. Then n satisfies (**) ,i.e., $|x_n - a| < \delta$. So, x_n satisfies (*). Thus,

$$|f(x) - f(a)| < \varepsilon.$$

Therefore, $f(x_n) \to f(a)$ as $n \to \infty$. □

5. Let $f(x) = x^2$. Prove that f is continuous on $\mathbb{R}$.

Proof. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{1+2|a|}\}$ such that $|x-a| < \delta$. Then $|x-a| < 1$. So, $|x| - |a| < |x-a| < 1$ or $|x| < 1 + |a|$.
We obtain

$$\begin{aligned}|f(x) - f(a)| &= |x^2 - a^2| = |(x-a)(x+a)| = |x-a||x+a| \\ &< \delta(|x| + |a|) < \delta(1 + 2|a|) \\ &< \frac{\varepsilon}{1+2|a|} \cdot (1 + 2|a|) = \varepsilon.\end{aligned}$$

Therefore, f is continuous on $\mathbb{R}$. □

6. Use IVT to prove that $\ln x = 3 - 2x$ has **at least one real root** by using caculator to **find an interval** $[a, b]$ **of length 0.01** (the length of $[a, b]$ means $b - a$) that contain a root.

Solution. Let $f(x) = \ln x + 2x - 3$. Consider each values of $f(x)$ by calculator

x	$f(x)$	Interval	Length of Interval
2	1.6931		
1	-1	$[1, 2]$	1
1.4	0.1365		
1.3	-0.1376	$[1.3, 1.4]$	0.1
1.35	0.00010		
1.34	-0.02733	$[1.34, 1.35]$	0.01

Since f is continuous on $(1.34, 1.35)$, we obtain that there is an $c \in (1.34, 1.35)$ such that

$$\ln c + 2c - 3 = f(c) = 0.$$

Thus, there exists a real number c such that $\ln c = 3 - 2c$.
We may approximate the root by choosing midpoint $c = 1.345$ of $(1.34, 1.35)$. It follows that $f(c) = -0.0136$ which has error 0.01.

7. Show that

$$f(x) = x^2 - x$$

is uniformly continuous on $(0, 1)$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$
Supose $x, a \in (0, 1)$ and $|x - a| < \delta$. Then $|x + a| < 2$. We obtain

$$\begin{aligned}
|f(x) - f(a)| &= |(x^2 - x) - (a^2 - a)| \\
&= |(x^2 - a^2) - (x - a)| \\
&= |(x - a)(x + a) - (x - a)| \\
&= |(x - a)(x + a - 1)| \\
&= |x - a||x + a - 1| \\
&< \delta(|x + a| + 1) \\
&< \delta \cdot 3 \\
&< \varepsilon
\end{aligned}$$

□

8. Show that

$$f(x) = \frac{1}{1 + x^2}$$

is uniformly continuous on $\mathbb{R}$. (Hint: Use the fact that $(|x| - 1)^2 \geq 0$ for all $x \in \mathbb{R}$)

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$
Supose $x, a \in \mathbb{R}$ and $|x - a| < \delta$. We obtain

$$\begin{aligned}
|f(x) - f(a)| &= \left| \frac{1}{1 + x^2} - \frac{1}{1 + a^2} \right| \\
&= \left| \frac{a^2 - x^2}{(1 + x^2)(1 + a^2)} \right| \\
&= \left| \frac{(x - a)(x + a)}{(1 + x^2)(1 + a^2)} \right| \\
&\leq |x - a| \cdot \frac{|x| + |a|}{(1 + x^2)(1 + a^2)} \\
&= |x - a| \left(\frac{|x|}{(1 + x^2)(1 + a^2)} + \frac{|a|}{(1 + x^2)(1 + a^2)} \right) \\
&\leq |x - a| \left(\frac{1}{2(1 + a^2)} + \frac{1}{2(1 + x^2)} \right) && 2|x| \leq x^2 + 1 \text{ and } 2|a| \leq 1 + a^2 \\
&\leq |x - a| \left(\frac{1}{2} + \frac{1}{2} \right) && 1 + x^2 > 1 \text{ and } 1 + a^2 > 1 \\
&= |x - a| \\
&< \delta = \varepsilon
\end{aligned}$$

□

9. Let $f : I \to \mathbb{R}$ where I is open. Assume that f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$. Prove that there are positive numbers ε and δ such that

$$|x - x_0| < \delta \quad \text{implies} \quad f(x) > \varepsilon.$$

Proof. Assume that f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$.
Given $\varepsilon = \dfrac{f(x_0)}{2}$. There is a $\delta > 0$ such that

$$|x - x_0| < \delta \text{ and } x \in I \quad \text{imply} \quad |f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

It follows that

$$\begin{array}{rcccl} -\dfrac{f(x_0)}{2} & < & f(x) - f(x_0) & < & \dfrac{f(x_0)}{2} \\ \dfrac{f(x_0)}{2} & < & f(x) & < & \dfrac{3f(x_0)}{2} \end{array}$$

Thus, $f(x) > \dfrac{f(x_0)}{2} = \varepsilon$. □

10. Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. Assume that f is differentiable at a and g is differentiable at $f(a)$.
Then $f'(a)$ and $g'(f(a))$ exist. We consider

$$\begin{aligned} f(x) &= \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a), \qquad x \neq a \\ g(y) &= \frac{g(y) - g(f(a))}{y - f(a)} \cdot (y - f(a)) + g(f(a)), \qquad y \neq f(a) \end{aligned} \tag{0.1}$$

Since f is continuous at a, substitue $y = f(x)$ in (0.1) to write

$$\begin{aligned} g(f(x)) &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot (f(x) - f(a)) + g(f(a)) \\ g(f(x)) &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \cdot (x - a) + g(f(a)) \\ \frac{g(f(x)) - g(f(a))}{x - a} &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ (g \circ f)'(a) &= g'(f(a)) \cdot f'(a) \end{aligned}$$

□