Math SSRU

Solution Assignment 8
MAC3309 Mathematical Analysis

Topic Differentiability **Score** 10 marks
Time $10th$ Week
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1. Show that $f(x) = x|x|$ is diferentiable on $\mathbb{R}$.

   Solution. By definition of absolute values,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

   It is clear that f is differentiable on $(-\infty, 0)$ and $(0, \infty)$ such that

$$f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$$

   Finall, we will prove that $f'(0) = 0$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x|x| - 0}{x} = \lim_{x\to 0} |x| = 0.$$

   Therefore, f is diferentiable on $\mathbb{R}$. and

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|$$

2. Show that the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

   is not differentiable at the origin.
   Hint: Use the SCL to show that the limit does not exist.

   Proof. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(\frac{1}{x})}{x} = \lim_{x\to 0} \sin(\frac{1}{x}).$$

   We will show that the limt does not exits by SCL.
   Let $g(x) = \sin(\frac{1}{x})$ where $x \neq 0$. Then $g(0)$ is undfined. Define two sequences

$$a_n = \frac{2}{(4n+1)\pi} \quad \text{where } n = 1, 2, 3, ...$$
$$b_n = \frac{2}{(4n+3)\pi} \quad \text{where } n = 1, 2, 3, ...$$

   Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$. Since a_n and b_n are non zero for all $n \in \mathbb{N}$,

$$g(a_n) = \sin\left(\frac{(4n+1)\pi}{2}\right) = 1$$
$$g(b_n) = \sin\left(\frac{(4n+3)\pi}{2}\right) = -1$$

   Hence, $\lim\limits_{n\to\infty} g(a_n) = 1 \neq -1 = \lim\limits_{n\to\infty} b(b_n)$. By SCL, we conclude that $\lim\limits_{x\to 0} \sin(\frac{1}{x})$ does not exist.

□

3. Apply L'Hospital's Rule to find $\lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right)$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right) &= \lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{-x^2}{1+x^2} \\
&= \lim_{x\to\infty} \frac{-1}{\frac{1}{x^2}+1} \\
&= \frac{-1}{1+0} = -1 \quad \#
\end{aligned}$$

4. Use the Mean Value Theorem to prove that

$$\sin x \leq x \quad \text{for all } x \geq 0$$

Proof. Let $a > 0$ and $f(x) = \sin x$ on $[0, a]$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. By the Mean Value Theorem (MVT), there is a $c \in (0, a)$ such that

$$\begin{aligned}
f(a) - f(0) &= f'(c)(a-0) \\
\sin a &= a\cos c
\end{aligned}$$

Since $\cos c \leq 1$

and $a > 0$, $a\cos c \leq a$. So, $\sin a \leq a$.
Therefore,

$$\sin x \leq x \quad \text{for all } x \geq 0.$$

□

5. Use the Mean Value Theorem to prove that

$$\cos x - 1 \leq x \quad \text{for all } x \geq 0$$

Proof. Let $a > 0$ and $f(x) = \cos x - 1$ on $[0, a]$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. By the Mean Value Theorem (MVT), there is a $c \in (0, a)$ such that

$$\begin{aligned}
f(a) - f(0) &= f'(c)(a-0) \\
\cos a - 1 - 0 &= (-\sin c)a
\end{aligned}$$

Since $-\sin c \leq 1$ and $a > 0$, $-a\sin c \leq a$. So, $\cos a - 1 \leq a$.
Therefore,

$$\cos x - 1 \leq x \quad \text{for all } x \geq 0$$

□

6. Find all $a \in \mathbb{R}$ such that

$$f(x) = ax^2 + 3x + 5$$

is strictly increasing on interval $(1, 2)$

Solution. We can find a by considering $f'(x) = 2ax + 3 \geq 0$ when $1 < x < 2$.
Then If $a = 0$, $f'(x) = 3 \geq 0$. It is done.
Case $a > 0$. Then $2a > 0$. So,

$$\begin{aligned} 2a \cdot 1 < 2a \cdot x < 2a \cdot 2 \\ 2a + 3 < 2ax + 3 < 4a + 3 \\ 2a + 3 < f'(x) < 4a + 3 \end{aligned}$$

Thus, $f'(x) > 0$ when $a > 0$.
Case $a < 0$. Then $2a < 0$

$$\begin{aligned} 2a \cdot 1 > 2a \cdot x > 2a \cdot 2 \\ 4a + 3 < 2ax < 2a + 3 \\ 4a + 3 < f'(x) < 2a + 3 \end{aligned}$$

We must be $4a + 3 \geq 0$. Thus, $-\frac{3}{4} < a < 0$.
Therefore, f is strictly increasing on interval $(1, 2)$ when $a > -\frac{3}{4}$.

7. Let $f(x) = x^2 e^{x^2}$ where $x \in \mathbb{R}$.

7.1 Use IFT to show that f^{-1} exists and its differentiable on $(0, \infty)$.

Proof. We see that f is continous on $\mathbb{R}$. It remains to show that f is 1-1 on $(0, \infty)$.
Let $x_1, x_2 \in (0, \infty)$ and $f(x_1) = f(x_2)$. Then

$$x_1^2 e^{x_1^2} = x_2^2 e^{x_2^2} \quad \longrightarrow \quad \frac{x_1^2}{x_2^2} \cdot e^{x_1^2 - x_2^2} = 1 \quad ...(*)$$

Suppose that $x_1 \neq x_2$. WLOG $x_1 > x_2$. Then $x_1^2 > x_2^2$ or $x_1^2 - x_2^2 > 0$. So

$$\frac{x_1^2}{x_2^2} > 1 \quad \text{and} \quad e^{x_1^2 - x_2^2} > 1$$

Thus, $\frac{x_1^2}{x_2^2} \cdot e^{x_1^2 - x_2^2} > 1$. This contradiction to $(*)$. Thus, $x_1 = x_2$.
Therefore, f^{-1} exists and its differentiable on $(0, \infty)$ by IFT. □

7.2 Compute $(f^{-1})'(e)$.
Solution. We see that $f'(x) = 2xe^{x^2} + 2x^3 e^{x^2}$ and $f(1) = e$. So $f^{-1}(e) = 1$. By IFT,

$$(f^{-1})'(e) = \frac{1}{f'(f^{-1}(e))} = \frac{1}{f'(1)} = \frac{1}{4e} \quad \#$$

8. Use the Inverse Function Theorem to prove that

$$(\arctan x)' = \frac{1}{1 + x^2} \qquad \text{for } x \in (-\infty, \infty)$$

Solution. Let $f(x) = \tan x$ when $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $f^{-1}(x) = \arctan x$ and $f'(x) = \sec^2 x$.
By the Inverse Function Theorem, we obatin

$$\begin{aligned} (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \\ (\arctan x)' &= \frac{1}{f'(\arctan x)} \\ &= \frac{1}{\sec^2(\arctan x)} \\ &= \frac{1}{1 + x^2} \end{aligned}$$