Solution Assignment 9
MAC3309 Mathematical Analysis

Topic Reimann Integral **Score** 10 marks
Time $11th$ Week
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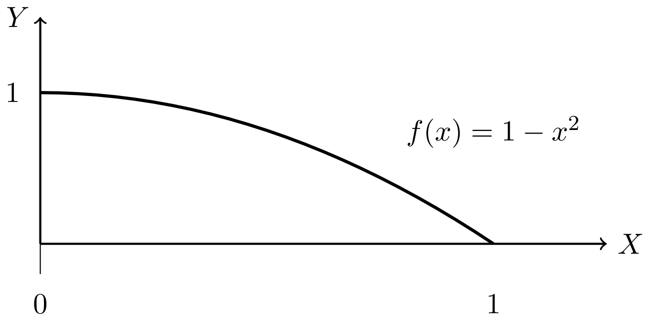
1. Let $f(x) = 1 - x^2$ on $[0, 1]$. Find $L(f, P)$ and $U(f, P)$ when $P = \left\{ \frac{j}{2^n} : j = 0, 1, 2, ..., 2^n \right\}$

   Solution.



Find $L(f, P)$. Consider $m_j(f) = f(\frac{j}{2^n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{2^n}$ for all $j = 1, 2, ..., 2^n$. We obtain

$$
\begin{aligned}
L(f,P) &= \sum_{j=1}^{2^n} m_j(f)\Delta x_j = \sum_{j=1}^{2^n} f\left(\frac{j}{2^n}\right)\frac{1}{2^n} \\
&= \frac{1}{2^n}\sum_{j=1}^{2^n}\left[1-\left(\frac{j}{2^n}\right)^2\right] = \frac{1}{2^n}\left[\sum_{j=1}^{2^n} 2^n - \sum_{j=1}^{2^n}\frac{1}{2^{2n}}\cdot j^2\right] \\
&= \frac{1}{2^n}\left[2^n - \frac{1}{2^{2n}}\sum_{j=1}^{2^n} j^2\right] = \frac{1}{2^n}\left[2^n - \frac{1}{2^{2n}}\left(\frac{2^n(2^n+1)(2\cdot 2^n+1)}{6}\right)\right] \\
&= 1 - \frac{2^n(2^n+1)(2^{n+1}+1)}{6\cdot 2^{3n}} \quad \#
\end{aligned}
$$

Find $U(f, P)$
Consider $M_j(f) = f(\frac{j-1}{2^n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., 2^n$.
We obtain

$$
\begin{aligned}
U(f,P) &= \sum_{j=1}^{2^n} M_j(f)\Delta x_j = \sum_{j=1}^{2^n} f\left(\frac{j-1}{2^n}\right)\frac{1}{2^n} \\
&= \frac{1}{2^n}\sum_{j=1}^{2^n}\left[1-\left(\frac{j-1}{2^n}\right)^2\right] = \frac{1}{2^n}\left[\sum_{j=1}^{2^n} 1 - \sum_{j=1}^{2^n}\frac{1}{2^{2n}}\cdot (j-1)^2\right] \\
&= \frac{1}{2^n}\left[2^n - \frac{1}{2^{2n}}[0^2+1^2+2^2+\cdots+(2^n-1)^2]\right] \\
&= \frac{1}{2^n}\left(2^n - \frac{1}{2^{2n}}\cdot\frac{(2^n-1)(2^n)(2(2^n-1)+1)}{6}\right) \\
&= 1 - \frac{(2^n-1)(2\cdot 2^n-1)}{6\cdot 2^{2n}} \quad \#
\end{aligned}
$$

2. Let $f(x) = 3x^2$ on $[0,1]$. Find $L(f,P)$ and $U(f,P)$ when

$$P = \left\{ \frac{j}{n} : j = 0, 1, 2, ..., n \right\}$$

Solution. Find $L(f,P)$

Consider $m_j(f) = f(\frac{j-1}{n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, ..., n$.
We obtain

$$\begin{aligned}
L(f,P) &= \sum_{j=1}^{n} m_j(f)\Delta x_j = \sum_{j=1}^{n} f\left(\frac{j-1}{n}\right)\frac{1}{n} \\
&= \frac{1}{n}\sum_{j=1}^{n} 3\left(\frac{j-1}{n}\right)^2 = \frac{3}{n}\sum_{j=1}^{n}\frac{1}{n^2}\cdot(j-1)^2 \\
&= \frac{3}{n^3}\sum_{j=1}^{n}(j-1)^2 = \frac{3}{n^3}[0^2 + 1^2 + 2^2 + \cdots + (n-1)^2] \\
&= \frac{3}{n^3}\left(\frac{(n-1)(n)(2(n-1)+1)}{6}\right) \\
&= \frac{(n-1)(2n-1)}{2n^2} \quad \#
\end{aligned}$$

Find $U(f,P)$

Consider $M_j(f) = f(\frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., n$.
We obtain

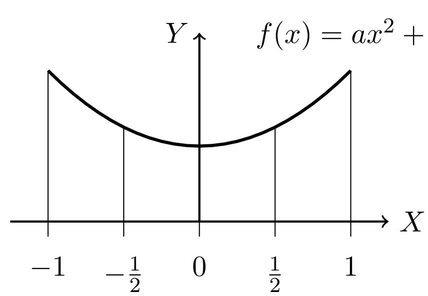
$$\begin{aligned}
U(f,P) &= \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} f\left(\frac{j}{n}\right)\frac{1}{n} \\
&= \frac{1}{n}\sum_{j=1}^{n} 3\left(\frac{j}{n}\right)^2 = \frac{3}{n}\sum_{j=1}^{n}\frac{1}{n^2}\cdot j^2 \\
&= \frac{3}{n^3}\sum_{j=1}^{n} j^2 = \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{(n+1)(2n+1)}{2n^2} \quad \#
\end{aligned}$$

3. Let $a > 0$ and $f(x) = ax^2 + 1$ where $x \in [-1, 1]$. Suppose that

$$U(f,P) - L(P,f) = 1 \quad \text{where} \quad P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}.$$

What is a ?

Solution. A graph of f is



Then

$$U(P,f) = \frac{1}{2}\left[f(-1) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(1)\right]$$
$$L(P,f) = \frac{1}{2}\left[f\left(-\frac{1}{2}\right) + f(0) + f(0) + f\left(\frac{1}{2}\right)\right]$$

We obtain

$$\begin{aligned}1 = U(P,f) - L(P,f) &= \frac{1}{2}\left[f(-1) + f(1) - 2f(0)\right] \\ &= \frac{1}{2}\left[(a+1) + (a+1) - 2(1)\right] \\ &= \frac{1}{2}(2a) = a\end{aligned}$$

It follows that $a = 1$. #

4. Let $f(x) = x^4$ where $x \in [0,1]$. Find

$$U(f,P) - L(P,f)$$

in term of n when

$$P = \left\{\frac{j}{n} : j = 0,1,2,...,n\right\}.$$

Solution. Let $x_j = \dfrac{j}{n}$ where $j = 0,1,2,...,n$. Consider the subinterval $[x_{j-1}, x_j]$, we get $\Delta x_j = \dfrac{1}{n}$ for all $j = 1,2,...,n$. Since f is increasing on $[0,1]$,

$$m_j(f) = f\left(\frac{j-1}{n}\right) \quad \text{and} \quad M_j(f) = f\left(\frac{j}{n}\right).$$

We obtain

$$\begin{aligned}U(f,P) &= \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} f\left(\frac{j}{n}\right)\frac{1}{n} \\ &= \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + f(1)\right] \\ L(f,P) &= \sum_{j=1}^{n} M_j(f)\Delta x_j = \sum_{j=1}^{n} f\left(\frac{j-1}{n}\right)\frac{1}{n} \\ &= \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right] \\ U(P,f) - L(P,f) &= \frac{1}{n}\left[f(1) - f(0)\right] = \frac{1}{n}(1-0) = \frac{1}{n}\end{aligned}$$

Hence,

$$U(f,P) - L(P,f) = \frac{1}{n}. \quad \#$$

5. Let f be integrable on $[a,b]$ and $f(x) \geq 0$. Prove that

$$\int_a^b f(x)\,dx = 0 \quad \text{if and only if} \quad f(x) = 0 \text{ (zero function)}$$

Proof. If $f(x) = 0$, then $m_j(f) = 0$ for all j. Thus, $L(f,P) = 0$ for all partition P. We conclude that

$$\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} = 0.$$

Assume that $\int_a^b f(x)\,dx = 0$. Then

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx = \sup\{U(f,P) : P \text{ is a partition of } [a,b]\} = 0$$

$$\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} = 0$$

We obtain

$$0 \leq L(f,P) \leq U(f,P) \leq 0$$

Thus, $U(f,P) = 0$ for any partition P of $[a,b]$. So,

$$0 = U(f,P) = \sum_{j=1}^{n} M_j(f)\Delta x_j$$

Since, $\Delta x_j > 0$ and $f(x) \geq 0$, $M_j(f) = 0$ for all j. We conclude that $f(x) = 0$ for all $x \in [a,b]$

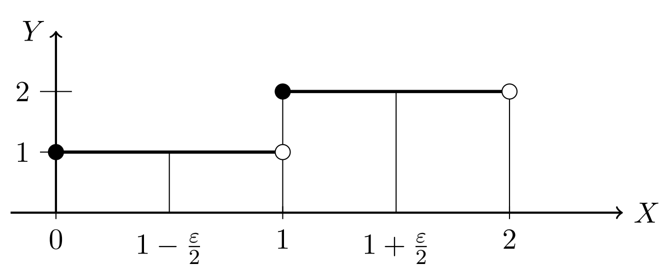
□

6. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x < 2 \end{cases}$$

Show that f is integrable on $[0,2]$

Solution. Let $\varepsilon > 0$. Case $\varepsilon \leq 1$. Choose $P = \left\{0, 1 - \frac{\varepsilon}{2}, 1, 1 + \frac{\varepsilon}{2}, 2\right\}$.



We obtain

$$\begin{aligned} U(f,P) &= 1\left(1 - \frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 2\left(1 - \frac{\varepsilon}{2}\right) \\ L(f,P) &= 1\left(1 - \frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 2\left(\frac{\varepsilon}{2}\right) + 2\left(1 - \frac{\varepsilon}{2}\right) \\ U(f,P) - L(f,P) &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Case $\varepsilon > 1$. Choose $P = \{0, 1, 2\}$. Then

$$\begin{aligned} U(f,P) &= 2\,(1-0) + 2\,(2-1) \\ L(f,P) &= 1\,(1-0) + 2\,(2-1) \\ U(f,P) - L(f,P) &= 1 < \varepsilon. \end{aligned}$$

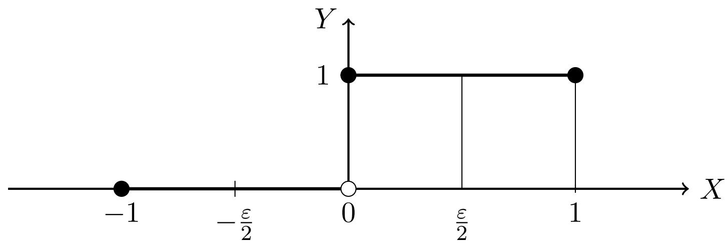
Thus, f is integrable on $[0,1]$.

7. Let

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 & \text{if } \quad 0 \le x \le 1 \end{cases}$$

Show that f is integrable on $[-1, 1]$

Solution. Let $\varepsilon > 0$. Case $\varepsilon \le 1$. Choose $P = \left\{-1, -\frac{\varepsilon}{2}, 0, \frac{\varepsilon}{2}, 1\right\}$.



We obtain

$$\begin{aligned} U(f,P) &= 0\left(1 - \frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(1 - \frac{\varepsilon}{2}\right) \\ L(f,P) &= 0\left(1 - \frac{\varepsilon}{2}\right) + 0\left(\frac{\varepsilon}{2}\right) + 1\left(\frac{\varepsilon}{2}\right) + 1\left(1 - \frac{\varepsilon}{2}\right) \\ U(f,P) - L(f,P) &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Case $\varepsilon > 1$. Choose $P = \{-1, 0, 1\}$. Then

$$\begin{aligned} U(f,P) &= 1\,(0 - (-1)) + 1\,(1 - 0) \\ L(f,P) &= 0\,(0 - (-1)) + 1\,(1 - 0) \\ U(f,P) - L(f,P) &= 1 < \varepsilon. \end{aligned}$$

Thus, f is integrable on $[0, 1]$.

8. Let $n \in \mathbb{N}$ and define $f : [0, n] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } \quad 0 \le x < 1 \\ 4 & \text{if } \quad 1 \le x < 2 \\ 9 & \text{if } \quad 2 \le x < 3 \\ \vdots & \qquad \vdots \\ n^2 & \text{if } \quad (n-1) \le x \le n \end{cases}$$

If $\displaystyle\int_0^n f(x)\,dx = 385$, what is n.

Solution. Consider

$$\begin{aligned} \int_0^n f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots + \int_{(n-1)}^n f(x)\,dx \\ &= \int_0^1 1\,dx + \int_1^2 4\,dx + \int_2^3 9\,dx + \cdots + \int_{(n-1)}^n n^2\,dx \\ &= 1^2 + 2^2 + 3^2 + \cdots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Then, $\dfrac{n(n+1)(2n+1)}{6} = 385$. That is

$$n(n+1)(2n+1) = 385 \cdot 6 = 11 \cdot 7 \cdot 5 \cdot 3 \cdot 2 = 10 \cdot 11 \cdot 21.$$

Therefore, $n = 10$. #