Math SSRU

Solution Quiz 1 : (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Ordered field axiom, Supremum & Infimume	**Score**	10 marks
Time	30 minutes (3*th* Week)	**Semester**	2/2023
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1. **(5 marks)** Let $x \in \mathbb{R}$ such that $0 < x < 1$. Prove that

$$x < \sqrt{x}.$$

Proof. Let $x \in \mathbb{R}$ such that $0 < x < 1$. Then $x > 0$. By O3.1, we have

$$x^2 = x \cdot x < 1 \cdot x = x.$$

We obtain $x^2 - x < 0$. It follows that

$$x^2 - (\sqrt{x})^2 < 0$$
$$(x - \sqrt{x})(x + \sqrt{x}) < 0.$$

Since $x + \sqrt{x} > 0$, $(x + \sqrt{x})^{-1} > 0$. By O3.1 again,

$$(x - \sqrt{x})(x + \sqrt{x})(x + \sqrt{x})^{-1} < 0 \cdot (x + \sqrt{x})^{-1}$$
$$x - \sqrt{x} < 0.$$

We conclude that $x < \sqrt{x}$.

□

2. **(5 marks)** Let $A = \left\{ \dfrac{2}{n+1} : n \in \mathbb{N} \right\}$. Find $\inf A$ and prove it.

We see that $A = \left\{ 1, \dfrac{2}{3}, \dfrac{2}{4}, \dfrac{2}{5}, \ldots \right\}$. **Claim that** $\inf A = 0$.

Proof. We will prove that $\inf A = 0$
Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $n + 1 > 0$. We obtain

$$\frac{2}{n+1} > 0$$

Thus, 0 is a lower bound of A.
Finally, we will show that 0 is the greatest lower bound of A.
Assume that that there is a lower bound ℓ_0 of A such that

$$\ell_0 > 0.$$

By definition,

$$\ell_0 \leq \frac{2}{n+1} \quad \text{for all } n \in \mathbb{N} \qquad (*)$$

From $\frac{\ell_0}{2} > 0$. By Archimendean property (2), there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\ell_0}{2} \qquad \longrightarrow \qquad \frac{2}{n_0} < \ell_0$$

Since $n_0 + 1 > n_0$,

$$\frac{2}{n_0+1} < \frac{2}{n_0} < \ell_0$$

This is contradiction to $(*)$. Therefore, $\inf A = 0$. $\square$

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Solution Quiz 1 : (1 p.m.)
MAC3309 Mathematical Analysis

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1. **(5 marks)** Let $x, y \in \mathbb{R}^+$. Prove that

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2.$$

Proof. Let $x, y \in \mathbb{R}^+$. In the fact that $(\sqrt{x} - \sqrt{y})^2 \geq 0$, we obtain

$$\begin{aligned} x - 2\sqrt{x}\sqrt{y} + y &\geq 0 \\ x + y &\geq 2\sqrt{x}\sqrt{y} \\ \frac{x}{\sqrt{x}\sqrt{y}} + \frac{y}{\sqrt{x}\sqrt{y}} &\geq 2 \\ \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} &\geq 2. \end{aligned}$$

□

2. **(5 marks)** Let $A = \left\{ \frac{2n}{n+1} : n \in \mathbb{N} \right\}$. Find **sup** $\boldsymbol{A}$ and prove it.

We see that $A = \left\{ 1, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \ldots \right\}$. **Claim that** $\sup A = 2$.

Proof. We will prove that $\sup A = 2$
Let $n \in \mathbb{N}$. Then $n \geq 1$. From $0 < 2$ So, $0 + 2n < 2 + 2n$. We obtain

$$\begin{aligned} 2n &< 2(n+1) \\ \frac{2n}{n+1} &< 2 \end{aligned}$$

Thus, 2 is an upper bound of A.
Finally, we will show that 2 is the least upper bound of A.
Assume that that there is an upper bound u_0 of A such that

$$u_0 < 2.$$

By definition,

$$\frac{2n}{n+1} \leq u_0 \quad \text{for all } n \in \mathbb{N} \qquad (*)$$

From $\frac{2 - u_0}{2} > 0$. By Archimendean property (2), there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2 - u_0}{2} \qquad \longrightarrow \qquad \frac{2}{n_0} < 2 - u_0$$

Since $n_0 + 1 > n_0$,

$$\frac{2}{n_0+1} < \frac{2}{n_0} < 2 - u_0$$
$$u_0 < 2 - \frac{2}{n_0+1} = \frac{2n_0}{n_0+1}.$$

This is contradiction to $(*)$. Therefore, $\sup A = 2$. $\square$