Math SSRU

Solution Quiz 3 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes (11*th* Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 2.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{24}\}$ such that $|x-2| < \delta$. Then $|x-2| < 1$.

$$\text{So, } |x| - |2| \leq |x-2| < 1. \text{ We obtain } |x| \leq 3.$$

By triangle inequality, it follows that

$$\begin{aligned} |f(x) - f(2)| &= |(x-1)(x-2)(x-3) - 0| \\ &= |x-1||x-2||x-3| \\ &< (|x|+1)\delta(|x|+3) \\ &< (3+1)\delta(3+3) \\ &= 24\delta < 24 \cdot \frac{\varepsilon}{24} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 2$. □

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x \leq x - 1 \quad \text{for all } x \geq 1.$$

Hints : Let $a > 1$ and consider function on $[1, a]$.

Proof. Let $a > 1$ and $f(x) = \ln x - x$ on $[1, a]$. Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. Then, $f'(x) = \dfrac{1}{x} - 1$. By the Mean Value Theorem (MVT), there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a-1) \\ (\ln a - a) - (0 - 1) &= \left(\frac{1}{c} - 1\right)(a-1) \\ (\ln a - a) + 1 &= \left(\frac{1-c}{c}\right)(a-1) \end{aligned}$$

From $1 < c < a$, $1 - c < 0$ and $a - 1 > 0$, we obtain

$$\left(\frac{1-c}{c}\right)(a-1) < 0.$$

So, $\ln a - a + 1 < 0$. Therefore,

$$\ln x \leq x - 1 \quad \text{for all } x \geq 0.$$

□

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Solution Quiz 3 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Continuity & the Mean Value Theorem (MVT)	**Score**	10 marks
Time	30 minutes ($11th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

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1. **(5 marks)** Let $f(x) = (x-1)(x-2)(x-3)$. Use the Definition to prove that

$$f \text{ is continuous at } 3.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{30}\}$ such that $|x-3| < \delta$. Then $|x-3| < 1$.

$$\text{So, } |x| - |3| \le |x-3| < 1. \text{ We obtain } |x| \le 4.$$

By triangle inequility, it follows that

$$\begin{aligned} |f(x) - f(3)| &= |(x-1)(x-2)(x-3) - 0| \\ &= |x-1||x-2||x-3| \\ &< (|x|+1)(|x|+2|)\delta \\ &< (4+1)(4+2)\delta \\ &= 30\delta < 30 \cdot \frac{\varepsilon}{30} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 3$. □

2. **(5 marks)** Use the Mean Value Theorem (MVT) to prove that

$$\ln x < x \quad \text{for all } x \ge 1.$$

Hints : Let $a > 1$ and consider function on $[1, a]$.

Proof. Let $a > 1$ and $f(x) = \ln x - x$ on $[1, a]$. Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. Then, $f'(x) = \dfrac{1}{x} - 1$. By the Mean Value Theorem (MVT), there is a $c \in (1, a)$ such that

$$\begin{aligned} f(a) - f(1) &= f'(c)(a-1) \\ (\ln a - a) - (0 - 1) &= \left(\frac{1}{c} - 1\right)(a-1) \\ (\ln a - a) + 1 &= \left(\frac{1-c}{c}\right)(a-1) \end{aligned}$$

From $1 < c < a$, $1 - c < 0$ and $a - 1 > 0$, we obtain

$$\left(\frac{1-c}{c}\right)(a-1) < 0.$$

So, $\ln a - a + 1 < 0$. It follows that $\ln a - a < -1 < 0$.
Therefore,

$$\ln x < x \quad \text{for all } x \ge 0.$$

□