Math SSRU

Solution Quiz 4 (8 a.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes (13th Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let $f(x) = 6x(x-1)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \frac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \frac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 6 \cdot \frac{j}{n}\left(\frac{j}{n} - 1\right) \\
&= \frac{6}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2} - \frac{j}{n}\right) = \frac{6}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 - \frac{1}{n}\sum_{j=1}^{n} j\right] \\
&= \frac{6}{n}\left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] \\
&= \frac{(n+1)(2n+1)}{n^2} - \frac{3(n+1)}{n}
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{(n+1)(2n+1)}{n^2} - \frac{3(n+1)}{n} = 2 - 3 = -1 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\int_{-1}^{0} f(x)\,dx = 67$. Use the change variable to compute

$$\int_{1}^{e} f(x\ln x - x) \cdot \ln x^2\,dx.$$

Solution. Let $\phi(x) = x\ln x - x$. Then $\phi'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x$,

$$\phi(1) = 1\ln 1 - 1 = 0 - 1 = -1 \quad \text{and} \quad \phi(e) = e\ln e - e = e - e = 0.$$

By the change variable, we obtain

$$\begin{aligned}
\int_{1}^{e} f(x\ln x - x) \cdot \ln x^2\,dx &= \int_{1}^{e} f(\phi(x)) \cdot 2\ln x\,dx \\
&= 2\int_{1}^{e} f(\phi(x)) \cdot \phi'(x)\,dx \\
&= 2\int_{\phi(1)}^{\phi(e)} f(t)\,dt \\
&= 2\int_{-1}^{0} f(t)\,dt = 5 \cdot 67 = 134 \quad \#
\end{aligned}$$

Math SSRU

Solution Quiz 4 (1 p.m.)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let $f(x) = 3x(x+2)$ where $x \in [0,1]$ and
$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$
be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \dfrac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \dfrac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be
$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 3 \cdot \frac{j}{n}\left(\frac{j}{n} + 2\right) \\
&= \frac{3}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2} + 2 \cdot \frac{j}{n}\right) = \frac{3}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 + \frac{2}{n}\sum_{j=1}^{n} j\right] \\
&= \frac{3}{n}\left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{2}{n} \cdot \frac{n(n+1)}{2}\right] \\
&= \frac{(n+1)(2n+1)}{2n^2} + \frac{3(n+1)}{n}
\end{aligned}$$
Thus,
$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n \to \infty} \frac{(n+1)(2n+1)}{2n^2} + \frac{3(n+1)}{n} = 1 + 3 = 4 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\displaystyle\int_0^1 f(x)\,dx = 67$. Use the change variable to compute
$$\int_0^1 f(e^x - xe^x) \cdot xe^x\,dx.$$

Solution. Let $\phi(x) = e^x - xe^x$. Then $\phi'(x) = e^x - (x \cdot e^x + 1 \cdot e^x) = -xe^x$,
$$\phi(0) = e^0 - 0e^0 = 1 = 1 - 0 = 1 \quad \text{and} \quad \phi(1) = e - e = 0.$$
By the change variable, we obtain
$$\begin{aligned}
\int_0^1 f(e^x - xe^x) \cdot xe^x\,dx &= -\int_0^1 f(\phi(x)) \cdot (-xe^x)\,dx \\
&= -\int_0^1 f(\phi(x)) \cdot \phi'(x)\,dx \\
&= -\int_{\phi(0)}^{\phi(1)} f(t)\,dt \\
&= -\int_1^0 f(t)\,dt = \int_0^1 f(t)\,dt = 67 \quad \#
\end{aligned}$$

Math SSRU

Solution Quiz 4 (Addition)
MAC3309 Mathematical Analysis

Topic	Riemann sum & Change variable	**Score**	10 marks
Time	30 minutes ($13th$ Week)	**Semester**	2/2023
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D.		
	Division of Mathematics, Faculty of Education,	Suan Sunandha Rajabhat University	

---

1. **(5 marks)** Let $f(x) = 6(x-1)(x+1)$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. Find the **Riemann Sum** of f and $\boldsymbol{I(f)}$.

Solution. Choose $t_j = \dfrac{j}{n}$ (the Right End Point) on the subinterval $[x_{j-1}, x_j]$ and $\Delta x_j = \dfrac{1}{n}$ for all $j = 1, 2, 3, ..., n$. We obtain the Riemann sum to be

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \cdot \frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n} 6\left(\frac{j}{n}-1\right)\left(\frac{j}{n}+1\right) \\
&= \frac{6}{n}\sum_{j=1}^{n}\left(\frac{j^2}{n^2}-1\right) = \frac{6}{n}\left[\frac{1}{n^2}\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} 1\right] \\
&= \frac{6}{n}\left[\frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - n\right] \\
&= \frac{(n+1)(2n+1)}{n^2} - 6
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} - 6 = 2 - 6 = -4 \quad \#$$

2. **(5 marks)** Let f be integrable $\mathbb{R}$ and $\displaystyle\int_1^{1+e} f(x)\,dx = 66$. Use the change variable to compute

$$\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{2x}\,dx.$$

Solution. Let $\phi(x) = \ln(xe^x) = \ln x + \ln e^x = \ln x + x$. Then $\phi'(x) = \frac{1}{x} + 1 = \frac{1+x}{x}$,

$$\phi(1) = \ln 1 + 1 = 0 + 1 = 1 \quad \text{and} \quad \phi(e) = \ln e + e = 1 + e.$$

By the change variable, we obtain

$$\begin{aligned}
\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{2x}\,dx &= \frac{1}{2}\int_1^e f(\ln(xe^x)) \cdot \frac{1+x}{x}\,dx \\
&= \frac{1}{2}\int_1^e f(\phi(x)) \cdot \phi'(x)\,dx \\
&= \frac{1}{2}\int_{\phi(1)}^{\phi(e)} f(t)\,dt \\
&= \frac{1}{2}\int_1^{1+e} f(t)\,dt = \frac{1}{2} \cdot 66 = 33 \quad \#
\end{aligned}$$