Math SSRU

Mathematical Analysis

Division of Mathematics Faculty of Education

Suan Sunandha Rajabhat University

2022

MATHEMATICAL ANALYSIS

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Update : November 2022

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Chapter 1

The Real Number System

1.1 Ordered field axioms

FIELD AXIOMS.

There are functions $+$ and $\cdot$, defined on $\mathbb{R}^2$, that satisfy the following properties for every $a, b, c \in \mathbb{R}$:

F1	**Closure Properties**	$a + b$ and $a \cdot b$ belong to $\mathbb{R}$.
F2	**Associative Properties**	$a + (b + c) = (a + b) + c$
		$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
F3	**Commutative Properties**	$a + b = b + a$ and $a \cdot b = b \cdot a$
F4	**Distributive Properties**	$a \cdot (b + c) = a \cdot b + a \cdot c$
		$(b + c) \cdot a = b \cdot a + c \cdot a$
F5	**Additive Identity**	There is a unique element $0 \in \mathbb{R}$ such that
		$0 + a = a = a + 0$ for all $a \in \mathbb{R}$.
F6	**Multiplicative Identity**	There is a unique element $1 \in \mathbb{R}$ such that
		$1 \cdot a = a = a \cdot 1$ for all $a \in \mathbb{R}$.
F7	**Additive Inverse**	For every $x \in \mathbb{R}$ there is a unique $-x \in \mathbb{R}$ such that
		$x + (-x) = 0 = (-x) + x$.
F8	**Multiplicative Inverse**	For every $x \in \mathbb{R}\backslash\{0\}$ there is a unique $x^{-1} \in \mathbb{R}$ such that
		$x \cdot (x^{-1}) = 1 = (x^{-1}) \cdot x$.

We shall frequently denote

$$a + (-b) \text{ by } a - b, \quad a \cdot b \text{ by } ab, \quad a^{-1} \text{ by } \frac{1}{a} \quad \text{and} \quad a \cdot b^{-1} \text{ by } \frac{a}{b}.$$

The real number system $\mathbb{R}$ contains certain special subsets: the set of **natural numbers**

$$\mathbb{N} := \{1, 2, 3, ...\}$$

obtained by begining with 1 and successively adding 1's to form $2 := 1 + 1$, $3 := 2 + 1$, etc,; the set of **integers**

$$\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$$

(Zahlen is German for number); the set of **rationals** (or fractions or quoteints)

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

and the set of **irrationals**

$$\mathbb{Q}^c := \mathbb{R} \backslash \mathbb{Q}.$$

Equality in $\mathbb{Q}$ is defined by

$$\frac{m}{n} = \frac{p}{q} \quad \text{if and only if} \quad mq = np.$$

Recall that each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ is a proper subset of the next; i.e.,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Definition 1.1.1 *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Define*

$$a^n = \underbrace{a \cdot a \cdot ... \cdot a}_{n- \ copies}$$

*a and n are called **base** and **exponent**, respectively.*

Definition 1.1.2 *Let a be a non-zero real number. Define*

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n} \quad \text{for } n \in \mathbb{N}$$

Theorem 1.1.3 *Let $a, b \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Then*

1. $(ab)^n = a^n b^n$
2. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ *where* $b \neq 0$
3. $a^n \cdot a^m = a^{m+n}$
4. $\frac{a^n}{a^m} = a^{n-m}$ *where* $a \neq 0$

Proof. Excercise. □

Theorem 1.1.4 *Let a be a real number. Then*

1. $0a = 0$
2. $(-1)a = -a$
3. $-(-a) = a$
4. $(a^{-1})^{-1} = a$ *where* $a \neq 0$

Theorem 1.1.5 *Let a and b be real numbers. Then*

$$-(ab) = a(-b) = (-a)b.$$

Theorem 1.1.6 (Cancellation) *Let a, b and c be real numbers. Then*

1. *Cancellation for addition* *if* $a + c = b + c$, *then* $a = b$.
2. *Cancellation for multiplication* *if* $ac = bc$ *and* $c \neq 0$, *then* $a = b$.

Theorem 1.1.7 (Integral Domain) *Let a and b be real numbers.*

$$\textit{If} \quad ab = 0 \ , \textit{ then} \quad a = 0 \quad \textit{or} \quad b = 0.$$

ORDER AXIOMS.

There is a relation $<$ on $\mathbb{R}^2$ that has the following properties for every $a, b, c \in \mathbb{R}$.

O1	**Trichotomy Property**	Given $a, b \in \mathbb{R}$, one and only one of the following statements holds: $a < b, \quad b < a, \quad \text{or} \quad a = b$
O2	**Trasitive Property**	$a < b$ and $b < c$ imply $a < c$
O3	**Additive Property**	$a < b$ imply $a + c < b + c$
O4	**Multiplicative Property**	O4.1 $a < b$ and $0 < c$ imply $ac < bc$ O4.2 $a < b$ and $c < 0$ imply $bc < ac$

We define in other cases:

- By $b > a$ we shall mean $a < b$.
- By $a \leq b$ we shall mean $a < b$ or $a = b$.
- If $a < b$ and $b < c$, we shall write $a < b < c$.
- We shall call a number $a \in \mathbb{R}$ **nonnegative** if $a \geq 0$ and **positive** if $a > 0$.

Example 1.1.8 *Let* $x \in \mathbb{R}$. *Show that if* $0 < x < 1$, *then* $0 < x^2 < x$

Example 1.1.9 *Let* $x, y \in \mathbb{R}$. *Show that if* $0 < x < y$, *then* $0 < x^2 < y^2$

Theorem 1.1.10 *Let* a, b, c *and* d *be real numbers.*

$$\text{If } a < b \text{ and } c < d, \text{ then } \quad a + c < b + d.$$

Theorem 1.1.11 *Let* a, b, c *and* d *be real numbers.*

$$\text{If } 0 < a < b \text{ and } 0 < c < d, \text{ then } \quad ac < bd.$$

Theorem 1.1.12 *If $a \in \mathbb{R}$, prove that*

$$a \neq 0 \quad \textit{implies} \quad a^2 > 0.$$

In particular, $-1 < 0 < 1$.

Example 1.1.13 *If $x \in \mathbb{R}$, prove that $x > 0$ implies $x^{-1} > 0$.*

Example 1.1.14 *If $x \in \mathbb{R}$, prove that $x < 0$ implies $x^{-1} < 0$.*

Theorem 1.1.15 *Let a and b be real numbers such that $0 < a < b$. Then*

$$\frac{1}{b} < \frac{1}{a}.$$

Example 1.1.16 *Let a and b be real numbers such that $b < a < 0$. Then*

$$\frac{1}{a} < \frac{1}{b}.$$

Example 1.1.17 *Let x and y be two distinct real numbers. Prove that*

$$\frac{x+y}{2} \text{ lies between } x \text{ and } y.$$

ABSOLUTE VALUE.

Definition 1.1.18 (Absolute Value) *The **absolute value** of a number $a \in \mathbb{R}$ is a the number*

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 1.1.19 (Positive Definite) *For all $a \in \mathbb{R}$,*

1. $|a| \geq 0$
2. $|a| = 0$ *if and only if* $a = 0$

Theorem 1.1.20 (Multiplicative Law) *For all $a, b \in \mathbb{R}$,*

$$|ab| = |a||b|.$$

Theorem 1.1.21 (**Symmetric Law**) *For all* $a, b \in \mathbb{R}$,

$$|a - b| = |b - a|.$$

Moreover, $|a| = |-a|$.

Example 1.1.22 *Show that* $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ *for all* $x \neq 0$.

Theorem 1.1.23 *Let $a, b \in \mathbb{R}$. Show that*

1. $|a^2| = a^2$
2. $a \leq |a|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ *when* $b \neq 0$

Theorem 1.1.24 *Let $a \in \mathbb{R}$ and $M \geq 0$. Then*

$$|a| \leq M \quad \text{if and only if} \quad -M \leq a \leq M$$

Corollary 1.1.25 *For all $a \in \mathbb{R}$,* $\quad -|a| \leq a \leq |a|$.

INTERVAL.

Let a and b real numbers. A **closed interval** is a set of the form

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} \qquad (-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$$

$$[a, \infty) := \{x \in \mathbb{R} : a \leq x\} \qquad (-\infty, \infty) := \mathbb{R},$$

and an **open interval** is a set of the form

$$(a, b) := \{x \in \mathbb{R} : a < x < b\} \qquad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

$$(a, \infty) := \{x \in \mathbb{R} : a < x\} \qquad (-\infty, \infty) := \mathbb{R}.$$

By an **interval** we mean a closed interval, an open interval, or a set of the form

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\} \quad \text{or} \quad (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Notice, then, that when $a < b$, then intervals $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) correspond to line segments on the real line, but when $b < a$, these interval are all the empty set.

Example 1.1.26 *Solve $|x - 1| \leq 1$ for $x \in \mathbb{R}$ in interval form.*

Example 1.1.27 *Show that if $|x| < 1$, then $|x^2 + x| < 2$.*

Example 1.1.28 *Show that if* $|x-1|<2$, *then* $\dfrac{1}{|x|}>1$.

Theorem 1.1.29 (**Triangle Inequality**) *Let* $a, b \in \mathbb{R}$. *Then,*

$$|a+b| \leq |a| + |b|.$$

Theorem 1.1.30 (Apply Triangle Inequality) *Let* $a, b \in \mathbb{R}$. *Then,*

1. $|a - b| \leq |a| + |b|$
2. $|a| - |b| \leq |a - b|$
3. $|a| - |b| \leq |a + b|$
4. $||a| - |b|| \leq |a - b|$

Example 1.1.31 *Show that if* $|x - 2| < 1$, *then* $|x| < 3$.

Theorem 1.1.32 *Let $x, y \in \mathbb{R}$. Then*

1. $x < y + \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \leq y$

2. $x > y - \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \geq y$

Corollary 1.1.33 *Let $a \in \mathbb{R}$. Then*

$$|a| < \varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{if and only if} \quad a = 0$$

Exercises 1.1

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1. Let $a, b \in \mathbb{R}$. Prove that

   1.1 $-(a-b) = b-a$

   1.2 $a(b-c) = ab-ac$

   1.3 $(-a)(-b) = ab$

   1.4 $\dfrac{-a}{b} = \dfrac{a}{-b} = -\dfrac{a}{b}$ when $b \neq 0$

2. Let $a, b \in \mathbb{R}$. Prove that

   2.1 If $a+b=a$, then $x=0$.

   2.2 If $ab=b$ and $b \neq 0$, then $a=1$.

   2.3 If $a^{-1}=a$ and $a \neq 0$, then $a=-1$ or $a=1$.

3. Let $a, b, c, d \in \mathbb{R}$. Prove that

   3.1 if $a < b < 0$, then $0 < b^2 < a^2$.

   3.2 if $a \leq b$ and $a \geq b$, then $a = b$.

   3.3 if $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

4. Solve each of the following inequality for $x \in \mathbb{R}$.

   4.1 $|1-2x| \leq 3$

   4.2 $|3-x| < 5$

   4.3 $|x^2-x-1| < x^2$

   4.4 $|x^2-x| < 2$

5. Prove that if $0 < a < 1$ and $b = 1-\sqrt{1-a}$, then $0 < b < a$.

6. Prove that if $a > 2$ and $b = 1-\sqrt{1-a}$, then $2 < b < a$.

7. Prove that $|x| \leq 1$ implies $|x^2-1| \leq 2|x-1|$.

8. Prove that $-1 \leq x \leq 2$ implies $|x^2+x-2| \leq 4|x-1|$.

9. Prove that $|x| \leq 1$ implies $|x^2-x-2| \leq 3|x+1|$.

10. Prove that $0 < |x-1| \leq 1$ implies $|x^3+x-2| < 8|x-1|$. Is this true if $0 \leq |x-1| < 1$?

11. Let $x, y \in \mathbb{R}$. Prove that if $|x+y| = |x-y|$, then $x|y| + y|x| = 0$.

12. Let $x, y \in \mathbb{R}$. Prove that if $|2x+y| = |x+2y|$, then $|xy| = x^2$.

13. Let $a \in \mathbb{R}$. Prove that $\dfrac{a^2+2}{\sqrt{a^2+1}} \geq 2$.

14. Prove that

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$

15. Let $x, y \in \mathbb{R}$. Prove that $x > y - \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.

16. Suppose that $x, a, y, b \in \mathbb{R}$, $|x-a| < \varepsilon$ and $|y-b| < \varepsilon$ for some $\varepsilon > 0$. Prove that

 16.1 $|xy - ab| < (|a| + |b|)\varepsilon + \varepsilon^2$

 16.2 $|x^2y - a^2b| < \varepsilon(|a|^2 + 2|ab|) + \varepsilon^2(|b| + 2|a|) + \varepsilon^2$

17. The **positive part** of an $a \in \mathbb{R}$ is defined by

$$a^+ := \frac{|a| + a}{2}$$

and the **negative part** by

$$a^- := \frac{|a| - a}{2}.$$

 17.1 Prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

 17.2 Prove that $a^+ := \begin{cases} a & : a \geq 0 \\ 0 & : a \leq 0 \end{cases}$ and $a^- := \begin{cases} 0 & : a \geq 0 \\ -a & : a \leq 0 \end{cases}$.

18. Let $a, b \in \mathbb{R}$. The **arithmetic mean** of a, b is $A(a,b) := \dfrac{a+b}{2}$,
the **geometric mean** of $a, b \in (0, \infty)$ is $G(a,b) := \sqrt{ab}$,
and **harmonic mean** of $a, b \in (0, \infty)$ is $H(a,b) := \dfrac{2}{a^{-1} + b^{-1}}$.
Show that

 18.1 if $a, b \in (0, \infty)$. Then $H(a,b) \leq G(a,b) \leq A(a,b)$.

 18.2 if $0 < a \leq b$. Then $a \leq G(a,b) \leq A(a,b) \leq b$.

 18.3 if $0 < a \leq b$. Then, $G(a,b) = A(a,b)$ if and only if $a = b$.

1.2 Well-Ordering Principle

Definition 1.2.1 *A number m is a* ***least element*** *of a set $S \subset \mathbb{R}$ if and only if*

$$m \in S \text{ and } m \leq s \text{ for all } s \in S.$$

WELL-ORDERING PRINCIPLE (WOP).

Every nonempty subset of $\mathbb{N}$ has a least element.

$$S \subseteq \mathbb{N} \wedge S \neq \varnothing \rightarrow \exists m \in S \, \forall s \in S, \; m \leq s.$$

Theorem 1.2.2 (**Mathematical Induction**) *Suppose for each $n \in \mathbb{N}$ that $P(n)$ is a statement that satisfies the following two properties:*

(1) Basic step : $P(1)$ is true

(2) Inductive step : For every $k \in \mathbb{N}$ for which $P(k)$ is true, $P(k+1)$ is also true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.2.3 (**Gauss' formula**) *Prove that*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Example 1.2.4 *Prove that* $2^n > n$ *for all* $n \in \mathbb{N}$.

BINOMIAL FORMULA.

Definition 1.2.5 *The notation* $0! = 1$ *and* $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ *for* $n \in \mathbb{N}$ *(called* ***factorial****), define the* ***binomial coefficient n over k*** *by*

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}$$

for $0 \leq k \leq n$ and $n = 0, 1, 2, 3, ...$

Theorem 1.2.6 *If* $n, k \in \mathbb{N}$ *and* $1 \leq k \leq n$*, then*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Theorem 1.2.7 (Binomial formula) *If* $a, b \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Exercises 1.2

1. Prove that the following formulas hold for all $n \in \mathbb{N}$.

   1.1 $\displaystyle\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n$

   1.2 $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

   1.3 $\displaystyle\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$

   1.4 $\displaystyle\sum_{k=1}^{n}\frac{a-1}{a^k} = 1 - \frac{1}{a^n}, \quad a \neq 0$

2. Use the Binomial Formula to prove each of the following.

   2.1 $2^n = \displaystyle\sum_{k=1}^{n}\binom{n}{k}$ for all $n \in \mathbb{N}$.

   2.2 $(a+b)^n \geq a^n + aa^{n-1}b$ for all $n \in \mathbb{N}$ and $a, b \geq 0$.

   2.3 $\left(1 + \dfrac{1}{n}\right)^n \geq 2$ for all $n \in \mathbb{N}$.

3. Let $n \in \mathbb{N}$. Write
$$\frac{(x+h)^n - x^n}{h}$$
as a sum none of whose terms has an h in the dennominator.

4. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_{n+1} < x_n < 1$ holds for all $n \in \mathbb{N}$.

5. Suppose that $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $2 \leq x_{n+1} \leq x_n \leq x_1$ holds for all $n \in \mathbb{N}$.

6. Suppose that $0 < x_1 < 2$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_n < x_{n+1} < 2$ holds for all $n \in \mathbb{N}$.

7. Prove that each of the following inequalities hold for all $n \in \mathbb{N}$.

   7.1 $n < 3^n$

   7.2 $n^2 \leq 2^n + 1$

   7.3 $n^3 \leq 3^n$

8. Let $0 < |a| < 1$. Prove that $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$.

9. Prove that $0 \leq a < b$ implies $a^n < b^n$ for all $n \in \mathbb{N}$.

1.3 Completeness Axiom

SUPREMUM.

Definition 1.3.1 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be **bounded above** if and only if*

$$\text{there is an } M \in \mathbb{R} \text{ such that } a \leq M \text{ for all } a \in A$$

2. *A number M is called an **upper bound** of the set A if and only if*

$$a \leq M \text{ for all } a \in A$$

3. *A number s is called a **supremum** of the set A if and only if*

$$s \text{ is an upper bound of } A \text{ and } s \leq M \text{ for all upper bound } M \text{ of } A$$

In this case we shall say that A has a supremum s and shall write $s = \sup A$

Example 1.3.2 *Fill the blanks of the following table.*

Sets	Bounded above	Set of Upper bound	Supremum
$A = [0, 1]$			
$A = (0, 1)$			
$A = \{1\}$			
$A = (0, \infty)$			
$A = (-\infty, 0)$			
$A = \mathbb{N}$			
$A = \mathbb{Z}$			

Example 1.3.3 *Show that* $\sup A = 1$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Theorem 1.3.4 *If a set has one upper bound, then it has infinitely many upper bounds.*

Theorem 1.3.5 *If a set has a supremum, then it has only one supremum.*

Theorem 1.3.6 (**Approximation Property for Supremum (APS)**) *If A has a supremum and $\varepsilon > 0$ is any positive number, then there is a point $a \in A$ such that*

$$\sup A - \varepsilon < a \leq \sup A$$

Theorem 1.3.7 *If* $A \subset \mathbb{N}$ *has a supremum, then* $\sup A \in A$.

COMPLETENESS AXIOM.

If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then A has a supremum.

Theorem 1.3.8 *The set of natural numbers is not bounded above.*

Theorem 1.3.9 (**Archimedean Properties (AP)**) *For each $x \in \mathbb{R}$, the following statements are true.*

1. *There is an integer $n \in \mathbb{N}$ such that $x < n$.*
2. *If $x > 0$, there there is an integer $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

Theorem 1.3.10 *Let $x \in \mathbb{R}$. Then*

$$|x| < \frac{1}{n} \quad \textit{for all } n \in \mathbb{N} \quad \textit{if and only if} \quad x = 0$$

Example 1.3.11 *Let* $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

Example 1.3.12 *Let* $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

INFIMUM.

Definition 1.3.13 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be* ***bounded below*** *if and only if*

$$\text{there is an } m \in \mathbb{R} \text{ such that } m \leq a \text{ for all } a \in A$$

2. *A number m is called a* ***lower bound*** *of the set A if and only if*

$$m \leq a \quad \text{for all } a \in A$$

3. *A number ℓ is called an* ***infimum*** *of the set A if and only if*

$$\ell \text{ is a lower bound of } A \text{ and } m \leq \ell \text{ for all lower bound } m \text{ of } A$$

In this case we shall say that A has an infimum s and shall write $\ell = \inf A$

4. *A is said to be* ***bounded*** *if and only if it is bounded above and below.*

Example 1.3.14 *Fill the blanks of the following table.*

Sets	Bounded below	Set of Lower bound	Infimum	Bounded
$A = [0, 1]$				
$A = (0, 1)$				
$A = \{1\}$				
$A = (0, \infty)$				
$A = (-\infty, 0)$				
$A = \mathbb{N}$				
$A = \mathbb{Z}$				

Example 1.3.15 *Show that* $\inf A = 0$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Example 1.3.16 *Let* $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = 0$.

Example 1.3.17 *Let* $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = \frac{1}{2}$.

Theorem 1.3.18 (**Approximation Property for Infimum (API)**) *If* A *has an infimum and* $\varepsilon > 0$ *is any positive number, then there is a point* $a \in A$ *such that*

$$\inf A \leq a < \inf A + \varepsilon.$$

Exercises 1.3

1. Find the infimum and supremum of each the following sets.

   1.1 $A = [0, 2)$

   1.2 $A = \{4, 3, 1, 5\}$

   1.3 $A = \{x \in \mathbb{R} : |x - 1| < 2\}$

   1.4 $A = \{x \in \mathbb{R} : |x + 1| < 1\}$

   1.5 $A = \{1 + (-1)^n : n \in \mathbb{N}\}$

   1.6 $A = \left\{\frac{1}{n} - (-1)^n : n \in \mathbb{N}\right\}$

   1.7 $A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

   1.8 $A = \left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}$

   1.9 $A = \left\{\frac{n^2+n}{n^2+1} : n \in \mathbb{N}\right\}$

   1.10 $A = \left\{\frac{n(-1)^n+1}{n+2} : n \in \mathbb{N}\right\}$

2. Find $\inf A$ and $\sup A$ with proving them.

   2.1 $A = [-1, 1]$

   2.2 $A = (-1, 2]$

   2.3 $A = (-1, 0) \cup (1, 2)$

   2.4 $A = \{1, 2, 3\}$

   2.5 $A = \left\{\frac{n}{n+2} : n \in \mathbb{N}\right\}$

   2.6 $A = \left\{\frac{n-2}{n+2} : n \in \mathbb{N}\right\}$

   2.7 $A = \left\{\frac{n}{n^2+1} : n \in \mathbb{N}\right\}$

   2.8 $A = \{(-1)^n : n \in \mathbb{N}\}$

3. Let $A = \left\{1 - \frac{n}{n^2+2} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

4. Let $A = \left\{2 - \frac{n}{n^2+1} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

5. If a set has one lower bound, then it has infinitely many lower bounds.

6. Prove that if A is a nonempty bounded subset of $\mathbb{Z}$, then both $\sup A$ and $\inf A$ exist and belong to A.

7. Prove that for each $a \in \mathbb{R}$ and each $n \in \mathbb{N}$ there exists a rational r_n such that

$$|a - r_n| < \frac{1}{n}.$$

8. Show that a lower bound of a set need not be unique but the infimum of a given set A is unique.

9. Show that if A is a nonoempty subset of $\mathbb{R}$ that is bounded below, then A has a finite infimum.

10. Prove that if x is an upper bound of a set $A \subseteq \mathbb{R}$ and $x \in A$, then x is the supremum of A.

11. Suppose $E, A, B \subset \mathbb{R}$ and $E = A \cup B$. Prove that if E has a supremum and both A and B are nonempty, then Sup$A$ and $\sup B$ both exist, and $\sup E$ is one of the numbers Sup$A$ or $\sup B$.

12. (**Monotone Property**) Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$. Prove that

    12.1 if B has a supremum, then $\sup A \leq \sup B$

    12.2 if B has an infimum, then $\inf B \leq \inf A$

13. Define the **reflection** of a set $A \subseteq \mathbb{R}$ by

$$-A := \{-x : x \in A\}$$

Let $A \subseteq \mathbb{R}$ be nonempty. Prove that

13.1 A has a supremum if and only if $-A$ has and infimum, in which case

$$\inf(-A) = -\sup A.$$

13.2 A has an infimum if and only if $-A$ has and supremum, in which case

$$\sup(-A) = -\inf A.$$

1.4 Functions and Inverse functions

Review notation $f : X \to Y$ that means a fuction form X to Y, each $x \in X$ is assigned a unique $y = f(x) \in Y$, there is nothing that keeps two x's from being assigned to the same y, and nothing that say every $y \in Y$ corresponds to some $x \in X$, i.e., f is a fuction if and only if for each $(x_1, y_1), (x_2, y_2)$ belong to f,

$$\text{if } x_1 = x_2 \text{ , then } y_2 = y_2.$$

Definition 1.4.1 *Let f be a function from a set X into a set Y.*

1. *f is said to be **one-to-one (1-1)** on X if and only if*

$$x_1, x_2 \in X \textit{ and } f(x_1) = f(x_2) \textit{ imply } x_1 = x_2.$$

2. *f is said to take X **onto** Y if and only if*

$$\textit{for each } y \in Y \textit{ there is an } x \in X \textit{ such that } y = f(x).$$

Example 1.4.2 *Show that $f(x) = 2x + 1$ is 1-1 from $\mathbb{R}$ onto $\mathbb{R}$.*

Theorem 1.4.3 *Let X and Y be sets and $f : X \to Y$. Then f is 1-1 from X onto Y if and only if there is a unique function g from Y onto X that satisfies*

1. $f(g(y)) = y, \quad y \in Y$

   and

2. $g(f(x)) = x, \quad x \in X$

If f is 1-1 from a set X onto a set Y, we shall say that f has an **inverse function**. We shall call the function g given in Theorem 1.4.3 the **inverse** of f, and denote it by f^{-1}. Then

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Example 1.4.4 *Find inverse function of* $f(x) = 2x + 1$.

Example 1.4.5 *Let* $f(x) = e^x - e^{-x}$.

1. *Show that* f *is 1-1 from* $\mathbb{R}$ *onto* $\mathbb{R}$.

2. *Find a formula of* $f^{-1}(x)$.

Exercises 1.4

1. For each of the following, prove f is 1-1 from A onto A. Find a formula for f^{-1}.

   1.1 $f(x) = 3x - 7$: $A = \mathbb{R}$

   1.2 $f(x) = x^2 - 2x - 1$: $A = (1, \infty)$

   1.3 $f(x) = 3x - |x| + |x - 2|$: $A = \mathbb{R}$

   1.4 $f(x) = x|x|$: $A = \mathbb{R}$

   1.5 $f(x) = e^{\frac{1}{x}}$: $A = (0, \infty)$

   1.6 $f(x) = \tan x$: $A = (-\frac{\pi}{2}, \frac{\pi}{2})$

   1.7 $f(x) = \dfrac{x}{x^2 + 1}$: $A = [-1, 1]$

2. Let $f(x) = x^2 e^{x^2}$ where $x \in \mathbb{R}$. Show that f is 1-1 on $(0, \infty)$.

3. Suppose that A is finite and f is 1-1 from A onto B. Prove that B is finite.

4. Prove that there a fuction f that is 1-1 from $\{2, 4, 6, ...\}$ onto $\mathbb{N}$.

5. Prove that there a fuction f that is 1-1 from $\{1, 3, 5, ...\}$ onto $\mathbb{N}$.

6. Suppose that $n \in \mathbb{N}$ and $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$.

   6.1 Prove that ϕ is 1-1 if and only if ϕ in onto.

   6.2 Suppose that A is finite and $f : A \to A$. Prove that

   $$f \text{ is 1-1 on } A \text{ if and only if } f \text{ takes } A \text{ onto } A.$$

7. Let $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a 1-1 function. Show that $\displaystyle\sum_{x=1}^{n} f(x) = n!$.

Chapter 2

Sequences in $\mathbb{R}$

2.1 Limits of sequences

An **infinite sequence** (more briefly, a sequence) is a function whose domain in $\mathbb{N}$. A sequence f whose term are $x_n := f(n)$ will be defined by

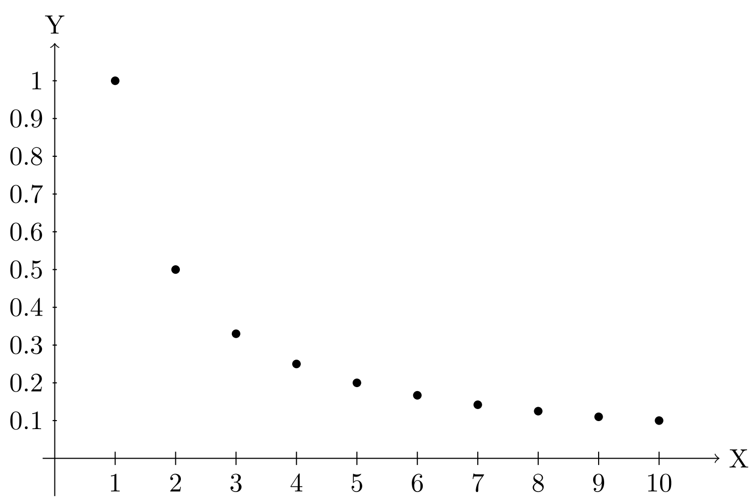
$$x_1, x_2, x_3, \ldots \quad \text{or} \quad \{x_n\}_{n\in\mathbb{N}} \quad \text{or} \quad \{x_n\}_{n=1}^{\infty} \quad \text{or} \quad \{x_n\}.$$

Example 2.1.1 *Use notation to represents the following sequences.*

1. $1, 2, 3, \ldots$ represents the sequence $\{n\}_{n\in\mathbb{N}}$

2. $1, -1, 1, -1, \ldots$ represents the sequence $\{(-1)^n\}$

Example 2.1.2 *Sketch graph of* $\{x_n\}$ *and guess* x_n *if* n *go to infinity where* $x_n = \dfrac{1}{n}$



Definition 2.1.3 *A sequence of real numbers $\{x_n\}$ is said to* ***converge*** *to a real number $a \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad |x_n - a| < \varepsilon.$$

We shall use the following phrases and notations interchangeably:

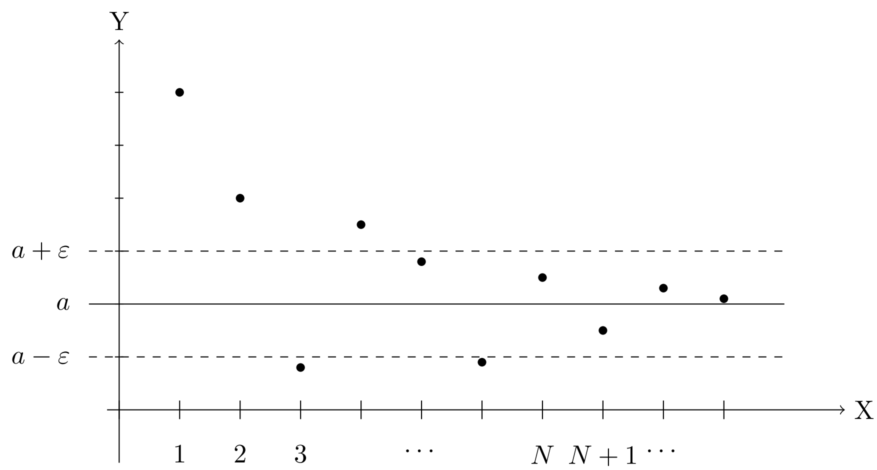
(a) $\{x_n\}$ converges to a;

(b) x_n converges to a;

(c) $\lim_{n\to\infty} x_n = a$;

(d) $x_n \to a$ as $n \to \infty$;

(e) the limit of $\{x_n\}$ exists and equals a.



Theorem 2.1.4 $\lim_{n\to\infty} k = k$ *where k is a constant.*

Example 2.1.5 *Prove that* $\dfrac{1}{n} \to 0$ *as* $n \to \infty$.

Example 2.1.6 *Prove that* $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$

Example 2.1.7 *Prove that* $\dfrac{1}{2^n} \to 0$ *as* $n \to \infty$

Example 2.1.8 *Prove that* $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2.1.9 *Prove that* $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

Example 2.1.10 *If* $x_n \to 1$ *as* $n \to \infty$. *Prove that*

$$2x_n + 1 \to 3 \text{ as } n \to \infty.$$

Example 2.1.11 *If* $x_n \to -1$ *as* $n \to \infty$. *Prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Example 2.1.12 *Assume that* $x_n \to 1$ *as* $n \to \infty$. *Show that*

$$\frac{1}{x_n} \to 1 \text{ as } n \to \infty.$$

Example 2.1.13 *Assume that $x_n \to 1$ as $n \to \infty$. Show that*

$$\frac{1 + (x_n)^2}{x_n + 1} \to 1 \text{ as } n \to \infty$$

Theorem 2.1.14 *A sequence can have at most one limit.*

Example 2.1.15 *Show that the limit* $\{(-1)^n\}_{n\in\mathbb{N}}$ *has no limit or does not exist (DNE).*

SUBSEQUENCES.

Definition 2.1.16 *By a **subsequence** of a sequence* $\{x_n\}_{n\in\mathbb{N}}$*, we shall mean a sequence of the form*

$$\{x_{n_k}\}_{k\in\mathbb{N}}, \quad \textit{where each } n_k \in \mathbb{N} \textit{ and } n_1 < n_2 < n_3 < \ldots$$

Example 2.1.17 *Give examples for two subsequences of the following sequences.*

Sequences	Subsequences
$1, -1, 1, -1, 1, -1, \ldots$	
$\{n\}_{n\in\mathbb{N}}$	

Theorem 2.1.18 *If* $\{x_n\}_{n\in\mathbb{N}}$ *converges to a and* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *is any subsequence of* $\{x_n\}_{n\in\mathbb{N}}$*, then*

$$x_{n_k} \textit{ converges to } a \textit{ as } k \to \infty.$$

Example 2.1.19 *Show that the limit* $\{\cos(n\pi)\}_{n\in\mathbb{N}}$ *has no limit.*

BOUNDED SEQUENCES.

Definition 2.1.20 *Let* $\{x_n\}$ *be a sequence of real numbers.*

1. $\{x_n\}$ *is said to be* ***bounded above*** *if and only if*

$$\text{there is an } M \in \mathbb{R} \text{ such that } \quad x_n \leq M \quad \text{for all } n \in \mathbb{N}$$

2. $\{x_n\}$ *is said to be* ***bounded below*** *if and only if*

$$\text{there is an } m \in \mathbb{R} \text{ such that } \quad m \leq x_n \quad \text{for all } n \in \mathbb{N}$$

3. $\{x_n\}$ *is said to be* ***bounded*** *if and only if it is both above and below or*

$$\text{there a } K > 0 \text{ such that } \quad |x_n| \leq K \quad \text{for all } n \in \mathbb{N}$$

Example 2.1.21 *Show that the following sequence is bounded above or bounded below or bounded.*

Sequences	Bounded below	Bounded above	Bounded
$\{n\}_{n\in\mathbb{N}}$			
$\{-n\}_{n\in\mathbb{N}}$			
$\{(-1)^n\}_{n\in\mathbb{N}}$			

Theorem 2.1.22 (**Bounded Convergent Theorem (BCT)**) *Every convergent sequence is bounded.*

Example 2.1.23 *Show that the limit* $\{n\}_{n\in\mathbb{N}}$ *does not exist.*

Example 2.1.24 *Assume that $x_n \to 1$ as $n \to \infty$. Use BCT to prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Exercises 2.1

1. Prove that the following limit exist.

1.1 $3+\frac{1}{n}$	as $n\to\infty$	1.5 $\frac{5+n}{n^2}$	as $n\to\infty$
1.2 $2\left(1-\frac{1}{n}\right)$	as $n\to\infty$	1.6 $\pi-\frac{3}{\sqrt{n}}$	as $n\to\infty$
1.3 $\frac{2n+1}{1-n}$	as $n\to\infty$	1.7 $\frac{n(n+2)}{n^2+1}$	as $n\to\infty$
1.4 $\frac{n^2-1}{n^2}$	as $n\to\infty$	1.8 $\frac{n}{n^3+1}$	as $n\to\infty$

2. Suppose that x_n is sequence of real numbers that converges to 2 as $n\to\infty$.

   Use Definition 2.1.3, prove that each of the following limit exists.

2.1 $2-x_n\to 0$	as $n\to\infty$	2.4 $\frac{1}{x_n-1}\to 1$	as $n\to\infty$
2.2 $3x_n+1\to 7$	as $n\to\infty$	2.5 $\frac{2+x_n^2}{x_n}\to 3$	as $n\to\infty$
2.3 $(x_n)^2+1\to 5$	as $n\to\infty$		

3. Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that $\lim\limits_{n\to\infty}(x_n-x_{n+1})=0$.

4. If $x_n\to a$ as $n\to\infty$, prove that $x_{n+1}\to a$ as $n\to\infty$.

5. If $x_n\to+\infty$ as $n\to\infty$, prove that $x_{n+1}\to+\infty$ as $n\to\infty$.

6. Prove that $\{(-1)^n\}$ has some subsequences that converge and others that do not converge.

7. Find a convergent subsequence of $n+(-1)^{3n}n$.

8. Suppose that $\{b_n\}$ is a sequence of nonnegative numbers that converges to 0, and $\{x_n\}$ is a real sequence that satisfies $|x_n-a|\le b_n$ for large n. Prove that x_n converges to a.

9. Suppose that $\{x_n\}$ is bounded. Prove that $\frac{x_n}{n^k}\to 0$ as $n\to\infty$ for all $k\in\mathbb{N}$.

10. Suppose that $\{x_n\}$ and $\{y_n\}$ converge to same point. Prove that $x_n-y_n\to 0$ as $n\to\infty$

11. Prove that $x_n\to a$ as $n\to\infty$ if and only if $x_n-a\to 0$ as $n\to\infty$.

2.2 Limits theorem

Theorem 2.2.1 (**Squeeze Theorem**) *Suppose that* $\{x_n\}$, $\{y_n\}$, *and* $\{w_n\}$ *are real sequences. If* $x_n \to a$ *and* $y_n \to a$ *as* $n \to \infty$, *and there is an* $N_0 \in \mathbb{N}$ *such that*

$$x_n \le w_n \le y_n \quad \text{for all } n \ge N_0,$$

then $w_n \to a$ *as* $n \to \infty$.

Example 2.2.2 *Use the Squeeze Theorem to prove that*

$$\lim_{n\to\infty} \frac{\sin(n^2)}{2^n} = 0.$$

Theorem 2.2.3 *Let* $\{x_n\}$, *and* $\{y_n\}$ *be real sequences. If* $x_n \to 0$ *and* $\{y_n\}$ *is bounded, then*

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Example 2.2.4 *Show that* $\lim_{n\to\infty} \frac{\cos(1+n)}{n^2} = 0.$

Theorem 2.2.5 *Let $A \subseteq \mathbb{R}$.*

1. *If A has a finite supremum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \sup A \quad \text{as} \quad n \to \infty.$$

2. *If A has a finite infimum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \inf A \quad \text{as} \quad n \to \infty.$$

Theorem 2.2.6 (**Additive Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are real sequences. If* $\{x_n\}$ *and* $\{y_n\}$ *are convergent, then*

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Theorem 2.2.7 (**Scalar Multiplicative Property**) *Let* $\alpha \in \mathbb{R}$. *If* $\{x_n\}$ *is a convergent sequence, then*

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n.$$

Theorem 2.2.8 (**Multiplicative Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. Then*

$$\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

Theorem 2.2.9 (**Reciprocal Property**) *Suppose that* $\{x_n\}$ *is a convergent sequence.*

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$$

where $\lim\limits_{n\to\infty} x_n \neq 0$ *and* $x_n \neq 0$.

Theorem 2.2.10 (**Quotient Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. Then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

where $\lim\limits_{n\to\infty} y_n \neq 0$ *and* $y_n \neq 0$.

Example 2.2.11 *Find the limit* $\lim_{n\to\infty} \dfrac{n^2+n-3}{1+3n^2}$.

Theorem 2.2.12 (**Comparison Theorem**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. If there is an* $N_0 \in \mathbb{N}$ *such that*

$$x_n \le y_n \quad \text{for all } n \ge N_0,$$

then

$$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a,b]$ *converges to some point* c*, then* c *must belong to* $[a,b]$.

DIVERGENT.

Definition 2.2.13 *Let $\{x_n\}$ be a sequence of real numbers.*

1. *$\{x_n\}$ is said to be* ***diverge*** *to $+\infty$, written $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n > M.$$

2. *$\{x_n\}$ is said to be* ***diverge*** *to $-\infty$, written $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n < M.$$

Example 2.2.14 *Show that* $\lim_{n\to\infty} n = +\infty$

Example 2.2.15 *Prove that* $\lim_{n\to\infty} \frac{n^2}{1+n} = +\infty.$

Example 2.2.16 *Prove that* $\lim_{n\to\infty} \dfrac{4n^2}{1-2n} = -\infty.$

Example 2.2.17 *Suppose that* $\{x_n\}$ *is a real sequence such that* $x_n \to +\infty$ *as* $n \to \infty$*. If* $x_n \neq 0$*, prove that*

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Theorem 2.2.18 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be a real sequence and* $x_n \neq 0$*. If* $\{y_n\}$ *is bounded and* $x_n \to +\infty$ *or* $x_n \to -\infty$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Example 2.2.19 *Show that* $\dfrac{\sin n}{n} \to 0$ *as* $n \to \infty$*.*

Theorem 2.2.20 *Let* $\{x_n\}$ *be a real sequence and* $\alpha > 0$.

1. *If* $x_n \to +\infty$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} (\alpha x_n) = +\infty$.
2. *If* $x_n \to -\infty$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} (\alpha x_n) = -\infty$.

Theorem 2.2.21 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be real sequences. Suppose that* $\{y_n\}$ *is bounded below and* $x_n \to +\infty$ *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty}(x_n + y_n) = +\infty.$$

Theorem 2.2.22 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be real sequences such that*

$$y_n > K \text{ for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

It follows that

1. *if* $x_n \to +\infty$ *as* $n \to \infty$, *then* $\lim_{n\to\infty}(x_n y_n) = +\infty$
2. *if* $x_n \to -\infty$ *as* $n \to \infty$, *then* $\lim_{n\to\infty}(x_n y_n) = -\infty$

Exercises 2.2

1. Prove that each of the following sequences coverges to zero.

   1.1 $x_n = \dfrac{\sin(n^4+n+1)}{n}$

   1.2 $x_n = \dfrac{n}{n^2+1}$

   1.3 $x_n = \dfrac{\sqrt{n}+1}{n+1}$

   1.4 $x_n = \dfrac{n}{2^n}$

   1.5 $x_n = \dfrac{(-1)^n}{n}$

   1.6 $x_n = \dfrac{1+(-1)^n}{2^n}$

2. Find the limit (if it exists) of each of the following sequences.

   2.1 $x_n = \dfrac{2n(n+1)}{n^2+1}$

   2.2 $x_n = \dfrac{1+n-3n^2}{3-2n+n^2}$

   2.3 $x_n = \dfrac{n^3+n+5}{5n^3+n-1}$

   2.4 $x_n = \dfrac{\sqrt{2n^2-1}}{n+1}$

   2.5 $x_n = \sqrt{n+2}-\sqrt{n}$

   2.6 $x_n = \sqrt{n^2+n}-n$

3. Prove that each of the following sequences coverges to $-\infty$ or $+\infty$.

   3.1 $x_n = n^2$

   3.2 $x_n = -n$

   3.3 $x_n = \dfrac{n}{1+\sqrt{n}}$

   3.4 $x_n = \dfrac{n^2+1}{n+1}$

   3.5 $x_n = \dfrac{1-n^2}{n}$

   3.6 $x_n = \dfrac{2^n}{n}$

4. Let $A \subseteq \mathbb{R}$. If A has a finite supremum, then there is a sequence $x_n \in A$ such that

$$x_n \to \sup A \quad \text{as} \quad n \to \infty.$$

5. Prove that given $x \in \mathbb{R}$ there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$.

6. Use the result Excercise 1.2, show that the following

   6.1 Suppose that $0 \leq x_1 \leq 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n \in \mathbb{N}$.

   If $x_n \to x$ as $n \to \infty$, prove that $x = 0$ or 1.

6.2 Suppose that $x_1 > 0$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$.

If $x_n \to x$ as $n \to \infty$, prove that $x = 2$.

7. Let $\{x_n\}$ be a real sequence and $\alpha > 0$. If $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = -\infty$.

8. Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that $y_n > K$ for some $K > 0$ and all $n \in \mathbb{N}$. Prove that if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$.

9. Let $\{x_n\}$ and $\{y_n\}$ are real sequences. Suuppose that $\{y_n\}$ is bounded above and $x_n \to -\infty$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} (x_n + y_n) = -\infty.$$

10. Interpret a decimal expansion $0.a_1a_2a_3...$ as

$$0.a_1a_2a_3... = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{a_k}{10^k}.$$

Prove that

10.1 $0.5 = 0.4999...$

10.2 $1 = 0.999...$

2.3 Bolzano-Weierstrass Theorem

MONOTONE.

Definition 2.3.1 *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers.*

1. $\{x_n\}$ *is said to be* ***increasing*** *if and only if* $x_1 \leq x_2 \leq x_3 \leq \ldots$ *or*

$$x_n \leq x_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

2. $\{x_n\}$ *is said to be* ***decreasing*** *if and only if* $x_1 \geq x_2 \geq x_3 \geq \ldots$ *or*

$$x_n \geq x_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

3. $\{x_n\}$ *is said to be* ***monotone*** *if and only if it is either increasing or decreasing.*

If $\{x_n\}$ is increasing and converges to a, we shall write $x_n \uparrow a$ as $n \to \infty$.

If $\{x_n\}$ is decreasing and converges to a, we shall write $x_n \downarrow a$ as $n \to \infty$.

Example 2.3.2 *Determine whether* $\{x_n\}_{n\in\mathbb{N}}$ *is increasing or decreasing or NOT both.*

Sequences	Decreasing	Increasing	Monotone
$\{n\}_{n\in\mathbb{N}}$			
$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$			
$\{1\}_{n\in\mathbb{N}}$			
$\{(-1)^n\}_{n\in\mathbb{N}}$			

Theorem 2.3.3 (**Monotone Converegence Theorem (MCT)**) *If* $\{x_n\}$ *is increasing and bounded above, or if it is decreasing and bounded below, then* $\{x_n\}$ *has a finite limit.*

Theorem 2.3.4 *If* $|a| < 1$, *then* $a^n \to 0$ *as* $n \to \infty$.

Example 2.3.5 *Find the limit of* $\left\{ \dfrac{3^{n+1} + 1}{3^n + 2^n} \right\}$.

Definition 2.3.6 *A sequence of sets* $\{I_n\}_{n\in\mathbb{N}}$ *is said to be* ***nested*** *if and only if*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \quad \text{or} \quad I_{n+1} \subseteq I_n \text{ for all } n \in \mathbb{N}.$$

Example 2.3.7 *Show that* $I_n = [\frac{1}{n}, 1]$ *is nested.*

Theorem 2.3.8 (Nested Interval Property) *If* $\{I_n\}_{n\in\mathbb{N}}$ *is a nested sequence of nonempty closed bounded intervals, then*

$$E = \bigcap_{n\in\mathbb{N}} I_n := \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

contains at least one number. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ *as* $n \to \infty$, *then* E *contains exactly one number.*

Theorem 2.3.9 (**Bolzano-Weierstrass Theorem**) *Every bounded sequence of real numbers has a convergence subsequence.*

Exercises 2.3

1. Prove that
$$x_n = \frac{(n^2+22n+65)\sin(n^3)}{n^2+n+1}$$
has a convergence sunsequence.

2. If $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ has a finite limit.

3. Suppose that $E \subset \mathbb{R}$ is nonempty bpunded set and $\sup E \notin E$. Prove that there exist a strictly increasing sequence $\{x_n\}$ ($x_1 < x_2 < x_3 < ...$) that converges to $\sup E$ such that $x_n \in E$ for all $n \in \mathbb{N}$.

4. Suppose that $\{x_n\}$ is a monotone increasing in $\mathbb{R}$ (not necessarily bounded above). Prove that there is extended real number x such that $x_n \to x$ as $n \to \infty$.

5. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Prove that
$$x_n \downarrow 0 \text{ as } n \to \infty \quad \text{and} \quad \frac{x_{n+1}}{x_n} \to \frac{1}{2}, \text{ as } n \to \infty$$

6. If $a > 0$, prove that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. Use the resulte to find the limit of $\{3^{\frac{n+1}{n}}\}$.

7. Let $0 \le x_1 \le 3$ and $x_{n+1} = \sqrt{2x_n + 3}$ for $n \in \mathbb{N}$. Prove that $x_n \uparrow 3$ as $n \to \infty$.

8. Suppose that $x_1 \ge 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $x_n \downarrow 2$ as $n \to \infty$. What happens when $1 \le x_1 < 2$?

9. Prove that
$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

10. Suppose that $x_0 \in \mathbb{R}$ and $x_n = \dfrac{1 + x_{n-1}}{2}$ for $n \in \mathbb{N}$. Prove that $x_n \to 1$ as $n \to \infty$.

11. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove that

    11.1 if $x_n \downarrow 0$, then $x_n > 0$ for all $n \in \mathbb{N}$.

11.2 if $x_n \uparrow 0$, then $x_n < 0$ for all $n \in \mathbb{N}$.

12. Let $0 < y_1 < x_1$ and set

$$x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n}, \quad \text{for } n \in \mathbb{N}$$

12.1 Prove that $0 < y_n < x_n$ for all $n \in \mathbb{N}$.

12.2 Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below.

12.3 Prove that $0 < x_{n+1} - y_{n+1} < \dfrac{x_1 - y_1}{2^n}$ for $n \in \mathbb{N}$

12.4 Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. (the common value is called the arithemetic-geometric mean of x_1 and y_1.)

13. Suppose that $x_0 = 1, y_0 = 0$

$$x_n = x_{n-1} + 2y_{n-1},$$

and

$$y_n = x_{n-1} + y_{n-1}$$

for $n \in \mathbb{N}$. Prove that $x_n^2 - 2y_n^2 = \pm 1$ for $n \in \mathbb{N}$ and

$$\frac{x_n}{y_n} \to \sqrt{2} \quad \text{as} \quad n \to \infty.$$

14. **(Archimedes)** Suppose that $x_0 = 2\sqrt{3}, y_0 = 3$,

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}}, \quad \text{and} \quad y_n = \sqrt{x_n y_{n-1}} \quad \text{for } n \in \mathbb{N}.$$

14.1 Prove that $x_n \downarrow x$ and $y_n \uparrow y$, as $n \to \infty$, for some $x, y \in \mathbb{R}$.

14.2 Prove that $x = y$ and

$$3.14155 < x < 3.14161.$$

(The actual value of x is π.)

2.4 Cauchy sequences

Definition 2.4.1 *A sequence of points $x_n \in \mathbb{R}$ is said to be* ***Cauchy*** *if and only if every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n, m \geq N \quad \textit{imply} \quad |x_n - x_m| < \varepsilon.$$

Example 2.4.2 *Show that $\left\{\dfrac{1}{n}\right\}$ is Cauchy.*

Example 2.4.3 *Show that $\left\{\dfrac{n}{n+1}\right\}$ is Cauchy.*

Theorem 2.4.4 *The sum of two Cauchy sequences is Cauchy.*

Theorem 2.4.5 *If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.*

Theorem 2.4.6 (Cauchy's Theorem) *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\{x_n\} \text{ is Cauchy if and only if } \{x_n\} \text{ converges to some point in } \mathbb{R}.$$

Example 2.4.7 *Prove that any real sequence* $\{x_n\}$ *that satisfies*

$$|x_n - x_{n+1}| \le \frac{1}{2^n}, \quad n \in \mathbb{N},$$

is convergent.

Exercises 2.4

1. Use definition to show that $\{x_n\}$ is Cauchy if

   1.1 $x_n = \dfrac{1}{n^2}$

   1.2 $x_n = \dfrac{n}{n+1}$

2. Prove that the product of two Cauchy sequences is Cauchy.

3. Prove that if $\{x_n\}$ is a sequence that satisfies

$$|x_n| \le \frac{1+n}{1+n+2n^2}$$

   for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.

4. Suppose that $x_n \in \mathbb{N}$ for $n \in \mathbb{N}$. If $\{x_n\}$ is Cauchy prove that there are numbers a and N such that $x_n = a$ for all $n \ge N$.

5. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement:

$$\text{if } n, m \ge N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

   Prove that $\{a_n\}$ converges.

   $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists and is finite.

6. Let $\{x_n\}$ be Cauchy. Prove that $\{x_n\}$ converges if and only if at least one of its subsequence converges.

7. Prove that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^k}{k}$ exists and is finite.

8. Let $\{x_n\}$ be a sequence. Suppose that there is an $a > 1$ such that

$$|x_{k+1} - x_k| \le a^{-k}$$

   for all $k \in \mathbb{N}$. Prove that $x_n \to x$ for some $x \in \mathbb{R}$.

9. Show that a sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Chapter 3

Topology on $\mathbb{R}$

3.1 Open sets

Open sets are among the most important subsets of $\mathbb{R}$. A collection of open sets is called a topology, and any property (such as convergence, compactness, or continuity) that can be dened entirely in terms of open sets is called a **topological property**.

Definition 3.1.1 *A set $E \subseteq \mathbb{R}$ is **open** if for every $x \in E$ there exists a $\delta > 0$ such that*

$$(x - \delta, x + \delta) \subseteq E.$$

In other word,

$$E \text{ is open} \quad \leftrightarrow \quad \forall x \in E\ \exists \delta > 0,\ (x - \delta, x + \delta) \subseteq E$$

$$\text{and}$$

$$E \text{ is not open} \quad \leftrightarrow \quad \exists x \in E\ \forall \delta > 0,\ (x - \delta, x + \delta) \nsubseteq E.$$

Since the empty set has no element, by definition it imples that $\varnothing$ is open. For $E = \mathbb{R}$, we obatin

$$\forall x \in \mathbb{R}\ \exists \delta > 0,\ (x - \delta, x + \delta) \subseteq \mathbb{R} \text{ is true.}$$

It follows that R is open.

Example 3.1.2 *Show that interval* $(0,1)$ *is open.*

Theorem 3.1.3 *Intervals* (a,b), (a,∞) *and* $(-\infty,b)$ *are open.*

Example 3.1.4 *Show that* $[0,1)$ *is not open.*

Theorem 3.1.5 *Let A and B be open. Prove that $A \cup B$ and $A \cap B$ are open.*

Theorem 3.1.6 *Let $A_1, A_2, .., A_n$ be open sets. Then*

1. $\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is open.*
2. $\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is open.*

NEIGHBORHOOD.

Next, we introduce the notion of the neighborhood of a point, which often gives clearer, but equivalent, descriptions of topological concepts than ones that use open intervals.

Definition 3.1.7 *A set $U \subseteq \mathbb{R}$ is a **neighborhood** of a point $x \in \mathbb{R}$ if*

$$(x-\delta, x+\delta) \subseteq U \quad \text{for some } \delta > 0.$$

For example $x = 1$, we have $(0, 2)$, $[0, 2]$ and $[0, 2)$ to be neighborhoods of 1.

Theorem 3.1.8 *A set $E \subseteq \mathbb{R}$ is open if every $x \in E$ has a neighborhood U such that $U \subseteq E$.*

Theorem 3.1.9 *A sequence $\{x_n\}$ of real numbers converges to a limit $x \in \mathbb{R}$ if and only if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.*

Exercises 3.1

1. Show that interval $[a, b]$, $[a, b)$ and $(a, b]$, are not open.

2. Show that interval $[a, \infty)$ and $(-\infty, b]$ are not open.

3. Give two neighborhoods of $x = 2$.

4. Let A and B be subsets of $\mathbb{R}$. Suppose that A and B are open.

   Determine whether $A \backslash B$ is open.

5. Let $U \subseteq \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

6. Let $A_1, A_2, .., A_n$ be open sets. Prove that

   6.1 $\displaystyle\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ is open.

   6.2 $\displaystyle\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ is open.

7. Find a sequence I_n of bounded, and open interval that

$$I_{n+1} \subset I_n \text{ for each } n \in \mathbb{N} \text{ and } \bigcap_{n=1}^{\infty} I_n = \varnothing.$$

3.2 Closed sets

Definition 3.2.1 *A set $F \subseteq \mathbb{R}$ is **closed** if*

$$F^c = \mathbb{R} \backslash F = \{x \in \mathbb{R} : x \notin F\} \text{ is open.}$$

Since $\varnothing^c = \mathbb{R}$ and $\mathbb{R}^c = \varnothing$ ($\varnothing$ and $\mathbb{R}$ are open), $\varnothing$ and $\mathbb{R}$ are closed sets.

Example 3.2.2 *Show that interval $[0, 1]$ is closed.*

Example 3.2.3 *Show that $[0, 1)$ is neither open nor closed.*

Theorem 3.2.4 *Let A and B be closed. Prove that $A \cup B$ and $A \cap B$ are closed.*

Theorem 3.2.5 *Let $A_1, A_2, .., A_n$ be closed sets. Then*

1. $\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is closed.*
2. $\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is closed.*

Exercises 3.2

1. Show that interval $[a,b]$, $[a,\infty)$ and $(-\infty,b]$ are closed.

2. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

3. Show that every closed interval I is a closed set.

4. Is $\displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right)$ open or closed ?

5. Is $\displaystyle\bigcup_{n=1}^{\infty}\left[\frac{1}{n},\frac{n-1}{n}\right]$ open or closed ?

6. Suppose, for $n \in \mathbb{N}$, the intervals $I_n = [a_n, b_n]$ are such that $I_{n+1} \subset I_n$. If

$$a = \sup\{a_n : n \in \mathbb{N}\} \quad \text{and} \quad b = \inf\{b_n : n \in \mathbb{N}\},$$

show that $\displaystyle\bigcap_{n=1}^{\infty} I_n = [a,b]$.

7. Find a sequence I_n of closed interval that $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$ and $\displaystyle\bigcap_{n=1}^{\infty} I_n = \varnothing$.

8. Suppose that $U \subseteq \mathbb{R}$ is a nonempty open set. For each $x \in U$, let

$$J_x = \big(x-\varepsilon, x+\delta\big),$$

where the union is taken over all $\varepsilon > 0$ and $\delta > 0$ such that $(x-\varepsilon, x+\delta) \subset U$.

8.1 Show that for every $x, y \in U$, either $J_x \cap J_y = \varnothing$, or $J_x = J_y$.

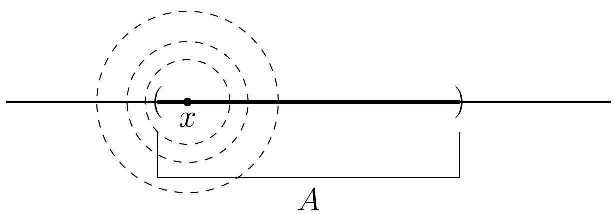
8.2 Show that $U = \displaystyle\bigcup_{x\in B} J_x$, where $B \subseteq U$ is either finite or countable.

3.3 Limit points

Definition 3.3.1 *A point $x \in \mathbb{R}$ is called a* ***limit point*** *of a set $A \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there exists $a \in A$, $a \neq x$, such that $a \in (x - \varepsilon, x + \varepsilon)$ or*

$$[(x - \varepsilon, x) \cup (x, x + \varepsilon)] \cap A \neq \varnothing.$$

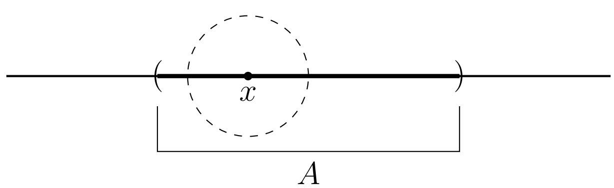
We denote the set of all limit points of a set A by A'.



Definition 3.3.2 *Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is an* ***interior point*** *of A if there exists an $\delta > 0$ such that*

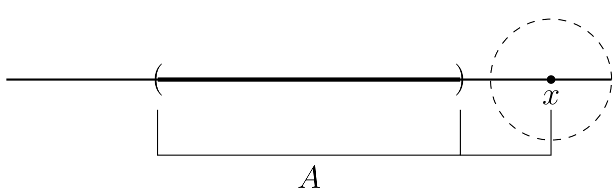
$$(x - \delta, x + \delta) \subseteq A.$$

The set of all interior points of A is called the interior of A, denoted A°.



Definition 3.3.3 *Suppose $A \subseteq \mathbb{R}$. A point $x \in A$ is called an* ***isolated point*** *of A if there exists an $\delta > 0$ such that*

$$A \cap (x - \delta, x + \delta) = \{x\}.$$



Example 3.3.4 *Fill the blanks of the following table.*

Set	Set of limit points	Set of interior points	Set of isolated points
$[0, 1]$			
$(0, 1)$			
$[0, 1)$			
$(0, 1] \cup \{3\}$			
$\{1\}$			
$\mathbb{N}$			
$\mathbb{Q}$			

Example 3.3.5 *Show that* 0 *is a limit point of* $(0, 1)$.

Theorem 3.3.6 *Let A and B be sets. If $A \subseteq B$, then $A' \subseteq B'$.*

Theorem 3.3.7 *Let A be a closed subset of $\mathbb{R}$. Then $A' \subseteq A$.*

CLOSURE.

Definition 3.3.8 *Given a set $A \subseteq R$, the set $\bar{A} = A \cup A'$ is called the **closure** of A.*

Example 3.3.9 *Fill the blanks of the following table.*

Set	Set of limit points	Closure
$[0, 1]$		
$(0, 1)$		
$[0, 1)$		
$(0, 1] \cup \{3\}$		
$\{1\}$		
$\mathbb{N}$		
$\mathbb{Q}$		

Theorem 3.3.10 *Let A and B be subsets of $\mathbb{R}$. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Theorem 3.3.11 *Let $A \subseteq \mathbb{R}$. Then $\bar{A}$ is closed.*

Theorem 3.3.12 *Let $A \subseteq \mathbb{R}$. Then A is closed if and only if $A = \bar{A}$.*

Theorem 3.3.13 *A set $F \subseteq \mathbb{R}$ is closed if and only if*

the limit of every convergent sequence in F belongs to F.

Exercises 3.3

1. Identify the limit points, interior point and isolated points of the following sets:

   1.1 $A = (0, 1) \cup \{3\}$

   1.2 $A = [0, 1]^c$

   1.3 $A = [1, \infty)$

   1.4 $A = (0, 1) \cup [3, 4]$

   1.5 $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

   1.6 $A = [0, 1] \cap \mathbb{Q}$

2. Find A', A° and $\bar{A}$ where

   2.1 $A = (0, 1)$

   2.2 $A = [0, 1]$

   2.3 $A = [0, \infty)$

   2.4 $A = (0, 1) \cup \{2, 3\}$

   2.5 $A = \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$

   2.6 $A = \mathbb{Q}$

3. Let A and B be two subset of $\mathbb{R}$. Show that $(A \cup B)' = A' \cup B'$.

4. Let A and B be two subset of $\mathbb{R}$. Determine whether

   4.1 $(A \cap B)' = A' \cap B'$

   4.2 $\overline{A \cup B} = \bar{A} \cup \bar{B}$

   4.3 $\overline{A \cap B} = \bar{A} \cap \bar{B}$

   4.4 $(A \cup B)^\circ = A^\circ \cup B^\circ$

   4.5 $(A \cap B)^\circ = A^\circ \cap B^\circ$

   4.6 if $\bar{A} \subseteq \bar{B}$, then $A \subseteq B$.

5. Prove that A° is open.

6. Prove that A is open if and only if $A = A^\circ$.

7. Suppose x is a limit point of the set A. Show that for every $\varepsilon > 0$, the set

$$(x - \varepsilon, x + \varepsilon) \cap A \text{ is infinite.}$$

8. Suppose that $A_k \subseteq \mathbb{R}$ for each $k \in \mathbb{N}$, and let $B = \bigcup_{k=1}^{\infty} A_k$. Show that $\bar{B} = \bigcup_{k=1}^{\infty} \bar{A}_k$.

9. If the limit of every convergent sequence in F belongs to $F \subseteq \mathbb{R}$, prove that F is closed.

Chapter 4

Limit of Functions

4.1 Limit of Functions

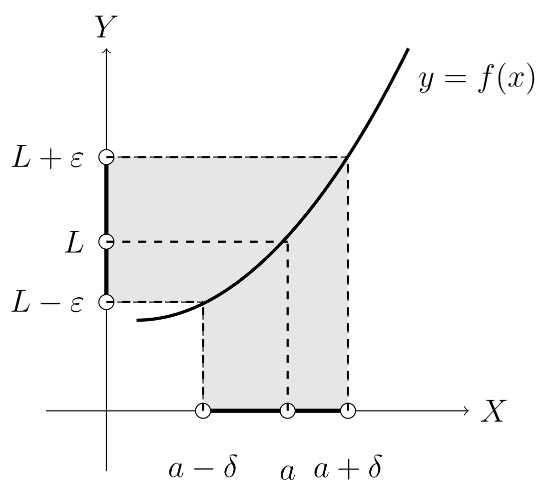
Definition 4.1.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a limit point of E. Then $f(x)$ is said to* ***converge*** *to L, as x* ***approaches*** *a, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in E$,*

$$0 < |x - a| < \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we write

$$\lim_{x \to a} f(x) = L \quad \textit{or} \quad f(x) \to L \textit{ as } x \to a.$$

and call L the ***limit*** *of $f(x)$ as x approaches a.*



Example 4.1.2 *Suppose that* $f(x) = 2x + 1$. *Prove that*

$$\lim_{x \to 1} f(x) = 3.$$

Example 4.1.3 *Let* $f(x) = \sqrt{x^2}$ *where* $x \in \mathbb{R}$. *Prove that* $f(x) \to 0$ *as* $x \to 0$.

Example 4.1.4 *Prove that*

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$$

Example 4.1.5 *Prove that*

$$\lim_{x\to 3} x^2 = 9.$$

Example 4.1.6 *Prove that* $f(x) = \dfrac{1}{x} \to 1$ *as* $x \to 1$.

Theorem 4.1.7 (**Limit of Constant function**) *The limit of a constant function is equal to the constant.*

Theorem 4.1.8 (**Limit of Linear function**) *Let m and c be constant such that $f(x) = mx + c$ for all $x \in \mathbb{R}$. Then*

$$\lim_{x \to a}(mx + c) = ma + c.$$

Theorem 4.1.9 *Let $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ be a limit point of E. If*

$$f(x) = g(x) \text{ for all } x \in E \backslash \{a\} \quad \text{and} \quad f(x) \to L \text{ as } x \to a,$$

then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Example 4.1.10 *Prove that $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a limit as $x \to 1$.*

Theorem 4.1.11 (**Sequential Characterization of Limit (SCL)**) *Let* $E \subseteq \mathbb{R}$ *and* $f : E \to \mathbb{R}$ *be a function and let* $a \in \mathbb{R}$ *be a limit point of* E*. Then*

$$\lim_{x \to a} f(x) = L \quad \text{exists}$$

if and only if $f(x_n) \to L$ *as* $n \to \infty$ *for every sequence* $x_n \in E \setminus \{a\}$ *that converges to* a *as* $n \to \infty$*.*

Example 4.1.12 *Use the SCL to prove that*

$$f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has no limit as $x \to 0$.

Example 4.1.13 *Use the SCL to prove that*

$$e^{-\frac{1}{x}} \to 0 \quad \text{as} \quad x \to 0^+.$$

Theorem 4.1.14 *Let $\alpha \in \mathbb{R}$, $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ be a limit point of E. If $f(x)$ and $g(x)$ converge as x approaches a, then so do*

$$(f+g)(x),\ (\alpha f)(x),\ (fg)(x) \text{ and } (\tfrac{f}{g})(x).$$

In fact,

1. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x)$
3. $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *when the limit of $g(x)$ is nonzero.*

Example 4.1.15 *Show that $\lim_{x\to a} x^2 = a^2$ fo all $a \in \mathbb{R}$.*

Theorem 4.1.16 *Suppose that* $E \subseteq \mathbb{R}$ *and* $f : E \to \mathbb{R}$ *is a function. Let* $a \in \mathbb{R}$ *be a limit point of* E*. Then,*

$$\lim_{x \to a} |f(x)| = 0 \quad \text{if and only if} \quad \lim_{x \to a} f(x) = 0.$$

Theorem 4.1.17 (**Squeeze Theorem for Functions**) *Suppose that* $E \subseteq \mathbb{R}$ *and* $f, g, h : E \to \mathbb{R}$ *are functions. Let* $a \in \mathbb{R}$ *be a limit point of* E*. If*

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in E \backslash \{a\},$$

and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$*, then the limit of* $f(x)$ *exists, as* $x \to a$ *and*

$$\lim_{x \to a} f(x) = L.$$

Corollary 4.1.18 *Suppose that $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ are functions. Let $a \in \mathbb{R}$ be a limit point of E and $M > 0$. If*

$$|g(x)| \leq M \quad \text{for all } x \in E \backslash \{a\} \quad \text{and} \quad \lim_{x \to a} f(x) = 0,$$

then

$$\lim_{x \to a} f(x)g(x) = 0.$$

Example 4.1.19 *Show that* $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$

Theorem 4.1.20 (**Comparision Theorem for Functions**) *Suppose that* $E \subseteq \mathbb{R}$ *and* $f, g : E \to \mathbb{R}$ *are functions. Let* $a \in \mathbb{R}$ *be a limit point of* E*. If* f *and* g *have a limit as* x *approaches* a *and*

$$f(x) \leq g(x), \quad x \in E \backslash \{a\},$$

then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Exercises 4.1

1. Use Definition 4.1.1, prove that each of the following limit exists.

   1.1 $\lim_{x\to 1} x^2 = 1$

   1.2 $\lim_{x\to 2} x^2 - x + 1 = 3$

   1.3 $\lim_{x\to -1} x^3 + 1 = 0.$

   1.4 $\lim_{x\to 0} \dfrac{x-1}{x+1} = -1$

2. Decide which of the following limit exist and which do not.

   2.1 $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$

   2.2 $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

   2.3 $\lim_{x\to 0} \tan\left(\dfrac{1}{x}\right)$

3. Evaluate the following limit using result from this section.

   3.1 $\lim_{x\to 1} \dfrac{x^2+x-2}{x^3-x}$

   3.2 $\lim_{x\to\sqrt{\pi}} \dfrac{\sqrt[3]{\pi - x^2}}{x+\pi}$

   3.3 $\lim_{x\to 0} x\sin\left(\dfrac{1}{x^2}\right)$

   3.4 $\lim_{x\to 0} x^2\cos\left(\dfrac{1}{x}\right)$

4. Prove that $\lim_{x\to 0} x^n \sin\left(\dfrac{1}{x}\right)$ exists for all $n \in \mathbb{N}$.

5. Show that $\lim_{x\to a} x^n = a^n$ fo all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

6. Prove that $\lim_{x\to a} |f(x)| = 0$ if and only if $\lim_{x\to a} f(x) = 0$.

7. Prove Squeeze Theorem for Functions.

8. Prove Comparision Theorem for Functions.

9. Suppose that f is a real function.

   9.1 Prove that if

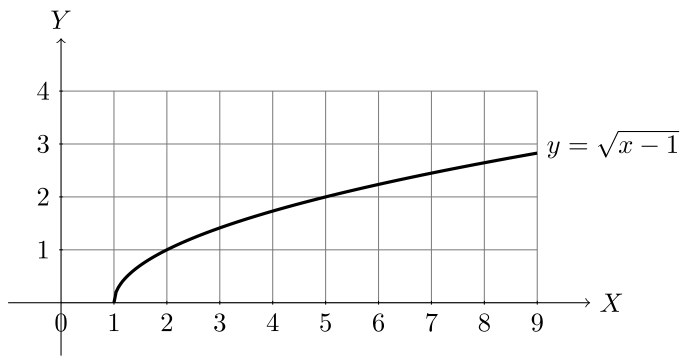
   $$\lim_{x\to a} f(x) = L$$

   exists, then $|f(x)| \to |L|$ as $x \to a$.

   9.2 Show that there is a function such that as $x \to a$, $|f(x)| \to |L|$ but the limit of $f(x)$ does not exist.

4.2 One-sided limit

What is the limit of $f(x) := \sqrt{x-1}$ as $x \to 1$.



A reasonable answer is that the limit is zero. This function, however, does not satisfy Definition 4.1.1 because it is not defined on an OPEN interval containg $a = 1$. Indeed, f is defined only for $x \geq 1$. To handle such situations, we introduce one-sided limits.

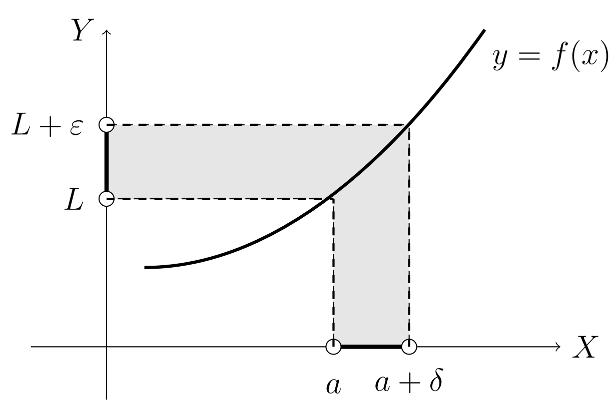
Definition 4.2.1 *Let $a \in \mathbb{R}$.*

1. *A real function f said to* ***converge*** *to L as x* ***approaches*** *a* ***from the right*** *if and only if f defined on some interval I with left endpoint a and every $\varepsilon > 0$ there is a $\delta > 0$ such that $a + \delta \in I$ and for all $x \in I$,*

$$a < x < a + \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we call L the ***right-hand limit*** *of f at a, and denote it by*

$$f(a^+) := L =: \lim_{x \to a^+} f(x).$$

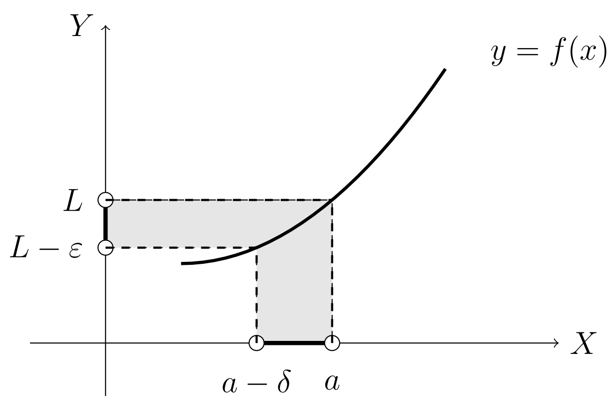


2. *A real function f said to* ***converge*** *to L as x* ***approaches*** *a* ***from the left*** *if and only if f defined on some interval I with right endpoint a and every $\varepsilon > 0$ there is a $\delta > 0$ such that $a + \delta \in I$ and for all $x \in I$,*

$$a - \delta < x < a \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we call L the ***left-hand limit*** *of f at a, and denote it by*

$$f(a^-) := L =: \lim_{x \to a^-} f(x).$$



Example 4.2.2 *Prove that*

1. $\lim_{x \to 1^+} \sqrt{x - 1} = 0$
2. $\lim_{x \to 0^-} \sqrt{-x} = 0$

Example 4.2.3 *If* $f(x) = \dfrac{|x|}{x}$, *prove that* f *has one-sided limit at* $a = 0$ *but* $\lim_{x \to 0} f(x) = 0$ *DNE.*

Theorem 4.2.4 *Let f be a real function. Then the limit*

$$\lim_{x \to a} f(x)$$

exists and equals to L if and only if

$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Example 4.2.5 *Use Theorem 4.2.4 to show that* $f(x) = \begin{cases} x + 1 & \textit{if } x \geq 0 \\ 2x + 1 & \textit{if } x < 0 \end{cases}$ *has limit at $a = 0$.*

Exercises 4.2

1. Use definitons to prove that $\lim_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = 2x^2 + 1,$ $\qquad a = 1$, and $L = 3$.

   1.2 $f(x) = \dfrac{x-1}{|1-x|},$ $\qquad a = 1$, and $L = 1$.

   1.3 $f(x) = \sqrt{3x-5},$ $\qquad a = 2$, and $L = 1$.

2. Use definitons to rove that $\lim_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = 1 + x^2,$ $\qquad a = 1$, and $L = 2$.

   2.2 $f(x) = \sqrt{1-x^2},$ $\qquad a = 1$, and $L = 0$.

   2.3 $f(x) = \dfrac{1-x^2}{1+x},$ $\qquad a = 1$, and $L = 0$.

3. Evauate the following limit when they exist.

   3.1 $\displaystyle\lim_{x\to 0^+} \frac{x+1}{x^2-2}$

   3.2 $\displaystyle\lim_{x\to 1^-} \frac{x^3-3x+2}{x^3-1}$

   3.3 $\displaystyle\lim_{x\to \pi^+} (x^2+1)\sin x$

   3.4 $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \frac{\cos x}{1-\sin x}$

4. Prove that $\dfrac{\sqrt{1-\cos x}}{\sin x} \to \dfrac{\sqrt{2}}{2}$ as $x \to 0^+$.

5. Determine whether the following functions are limit at a.

   5.1 $f(x) = \begin{cases} 3x+1 & \text{if } x \geq 1 \\ x+3 & \text{if } x < 1 \end{cases}$ and $\quad a = 1$

   5.2 $f(x) = \begin{cases} 2-2x & \text{if } x \geq 0 \\ \sqrt{1-x} & \text{if } x < 0 \end{cases}$ and $\quad a = 0$

6. Suppose that $f : [0,1] \to \mathbb{R}$ and $f(a) = \lim_{x\to a} f(x)$ for all $x \in [0,1]$. Prove that

$$f(q) = 0 \text{ for all } q \in \mathbb{Q} \cap [0,1] \text{ if and only if } f(x) = 0 \text{ for all } x \in [0,1].$$

4.3 Infinite limit

The definition of limit of real functions can be expanded to include extended real numbers.

Definition 4.3.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to L$ as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x > M \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

   In this case we shall write $\lim\limits_{x\to\infty} f(x) = L$.

2. *We say that $f(x) \to L$ as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(-\infty, -c) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x < M \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

   In this case we shall write $\lim\limits_{x\to-\infty} f(x) = L$.

Example 4.3.2 *Prove that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.*

Example 4.3.3 *Prove that* $\lim_{x\to\infty} \frac{x-1}{x+1}$ *exists and equals to 1.*

Example 4.3.4 *Prove that* $\lim_{x\to\infty} \frac{1}{x^2+1} = 0.$

Example 4.3.5 *Prove that* $\lim_{x \to -\infty} \frac{1}{x} = 0.$

Example 4.3.6 *Prove that* $\lim_{x \to -\infty} \frac{x}{x+1} = 1.$

Definition 4.3.7 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to +\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M > 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \text{implies} \quad f(x) > M.$$

   In this case we shall write $\lim\limits_{x \to a} f(x) = +\infty$.

2. *We say that $f(x) \to -\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M < 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \text{implies} \quad f(x) < M.$$

   In this case we shall write $\lim\limits_{x \to a} f(x) = -\infty$.

Obviousl modification define $f(x) \to \pm\infty$ as $x \to a^+$ and $x \to a^-$, and $f(x) \to \pm\infty$ as $x \to \pm\infty$.

Example 4.3.8 *Prove that* $\lim\limits_{x \to 0} \dfrac{1}{|x|} = +\infty$.

Example 4.3.9 *Prove that* $\lim\limits_{x \to 1^+} \dfrac{x}{1-x} = -\infty.$

Example 4.3.10 *Prove that* $\lim\limits_{x \to 1^-} \dfrac{x}{1-x} = +\infty.$

Exercises 4.3

1. Use definitons to prove that $\lim_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = \dfrac{1}{x-3}$, $a = 3$, and $L = +\infty$.

   1.2 $f(x) = -\dfrac{1}{x}$, $a = 0$, and $L = -\infty$.

2. Use definitons to prove that $\lim_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = \dfrac{x}{x^2-4}$, $a = 2$, and $L = -\infty$.

   2.2 $f(x) = \dfrac{1}{1-x^2}$, $a = 1$, and $L = +\infty$.

3. Use definition to prove that the follwing limits

   3.1 $\lim_{x\to\infty} \dfrac{2x+1}{x+1} = 2$

   3.2 $\lim_{x\to-\infty} \dfrac{1-x}{2x+1} = -\dfrac{1}{2}$

   3.3 $\lim_{x\to\infty} \dfrac{2x^2+1}{1-x^2} = -2$

   3.4 $\lim_{x\to 2} \dfrac{x}{|x-2|} = +\infty$

   3.5 $\lim_{x\to 2^+} \dfrac{x+1}{x-2} = +\infty$

   3.6 $\lim_{x\to 2^-} \dfrac{x+1}{x-2} = -\infty$

4. Evauate the following limit when they exist.

   4.1 $\lim_{x\to\infty} \dfrac{3x^2-13x+4}{1-x-x^2}$

   4.2 $\lim_{x\to\infty} \dfrac{x^2+x+2}{x^3-x-2}$

   4.3 $\lim_{x\to-\infty} \dfrac{x^3-1}{x^2+2}$

   4.4 $\lim_{x\to\infty} \arctan x$

   4.5 $\lim_{x\to\infty} \dfrac{\sin x}{x^2}$

   4.6 $\lim_{x\to-\infty} x^2 \sin x$

5. Prove that $\dfrac{\sin(x+3)-\sin 3}{x}$ converges to 0 as $x \to \infty$.

6. Prove the following comparision theorems for real functions.

   6.1 If $f(x) \geq g(x)$ and $g(x) \to \infty$ as $x \to a$, then $f(x) \to \infty$ as $x \to a$.

   6.2 If $f(x) \leq g(x) \leq h(x)$ and $L = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x)$, then $g(x) \to L$ as $x \to \infty$.

7. Recall that a **polynomial of degree n** is a functon of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_j \in \mathbb{R}$ for $j = 0, 1, ..., n$ and $a_n \neq 0$.

7.1 Prove that $\lim_{x \to a} x^n = a^n$ for $n = 0, 1, 2, ...$

7.2 Prove that if P is a polynomial, then

$$\lim_{x \to a} P(x) = P(a)$$

for every $a \in \mathbb{R}$.

7.3 Suppose that P is a polynomial and $P(a) > 0$. Prove that $\frac{P(x)}{x-a} \to \infty$ as $x \to a^+$, $\frac{P(x)}{x-a} \to -\infty$ as $x \to a^-$, but

$$\lim_{x \to a} \frac{P(x)}{x-a}$$

does not exist.

8. **Cauchy**. Suppose that $f : \mathbb{N} \to \mathbb{R}$. If

$$\lim_{n \to \infty} f(n+1) - f(n) = L,$$

prove that $\lim_{n \to \infty} \frac{f(n)}{n}$ exists and equals L.

Chapter 5

Continuity on $\mathbb{R}$

5.1 Continuity

Definition 5.1.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*
f is said to be ***continuous at a point*** *$a \in E$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \text{ and } x \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.1.2 *Let $f(x) = 2x - 1$ where $x \in \mathbb{R}$. Prove that f is continuous at $x = 1$.*

Example 5.1.3 *Let* $f(x) = x^2$ *where* $x \in \mathbb{R}$. *Prove that* f *is continuous at* $x = 2$.

Example 5.1.4 *Let* $f(x) = \sqrt{x}$ *where* $x \in (0, \infty)$. *Prove that* f *is continuous at 1.*

Example 5.1.5 *Let* $f(x) = 3 - x^2$ *where* $x \in [-1, 2] \cup \{3\}$. *Prove that* f *is continuous at* $x = 3$

Example 5.1.6 *Prove that the function*

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0.

Theorem 5.1.7 *Let I be an open interval that contain a point a and $f : I \to \mathbb{R}$. Then*

$$f \text{ is continuous at } a \in I \text{ if and only if } \quad f(a) = \lim_{x \to a} f(x).$$

Example 5.1.8 *Let* $f(x) = x\cos\left(\frac{1}{x}\right)$ *where* $x \neq 0$. *If* f *is continuous at 0, what is* $f(0)$ *defined?*

Example 5.1.9 *Find* a *such that the function* $f(x) = \begin{cases} ax+1 & \text{if } x \geq 1 \\ 2x+3 & \text{if } x < 1 \end{cases}$ *is continuous at 1.*

Theorem 5.1.10 *Suppose that E is a nonempty subset of $\mathbb{R}$, $a \in E$, and $f : E \to \mathbb{R}$. Then the following statements are equivalent:*

1. *f is continuous at $a \in E$.*

2. *If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.*

Example 5.1.11 *Use Theorem 5.1.10 to find $\lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$.*

Theorem 5.1.12 *Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. If f, g are continuous at a point $a \in E$, then so are*

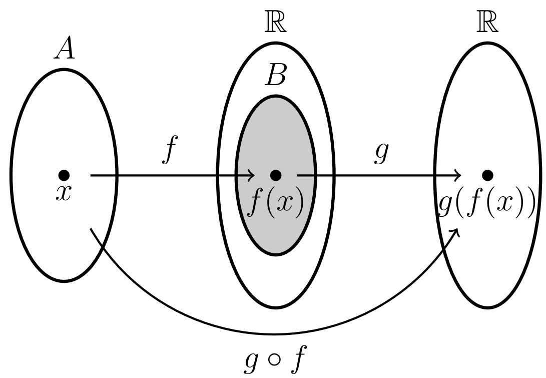
$$f + g, \quad fg \quad \text{and} \quad \alpha f$$

Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$.

CONTINUITY OF COMPOSITION.

Definition 5.1.13 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $\{f(x) : x \in A\} \subseteq B$, then the composition of g with f is the function*

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$



Theorem 5.1.14 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $\{f(x) : x \in A\} \subseteq B$. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then*

$$g \circ f \text{ is continuous at } a \in A$$

and moreover,

$$\lim_{x \to a} (g \circ f)(x) = g\left(\lim_{x \to a} f(x)\right).$$

Example 5.1.15 *Show that* $\lim_{x \to 1} \sqrt{2x-1}$ *exists and equals to 1.*

CONTINUITY ON A SET.

Definition 5.1.16 *Let* E *be a nonempty subset of* $\mathbb{R}$ *and* $f : E \to \mathbb{R}$*.*

> f *is said to be* ***continuous on E*** *if and only if* f *is continuous at every* $a \in E$*.*

Note that if f is continuous on E, then f is continuous on nonempty subset of E.

Example 5.1.17 *Show that* $f(x) = x^2$ *is continuous on* $\mathbb{R}$*.*

Theorem 5.1.18 (**Continuity of Linear function**) *Let m and c be constants and let*

$$f(x) = mx + c \textit{ where } x \in \mathbb{R}.$$

Prove that f is continuous on $\mathbb{R}$

Example 5.1.19 *Show that $h(x) = (3x+1)^2$ is continuous on $\mathbb{R}$.*

Example 5.1.20 *Prove that*

$$f(x) = \begin{cases} 2x+4 & \text{if } x > -1 \\ 3x+5 & \text{if } x \le -1 \end{cases}$$

is continuous on $\mathbb{R}$.

Example 5.1.21 *Find* a *such that the function* $f(x) = \begin{cases} ax+1 & \text{if } x \ge 2 \\ x+a & \text{if } x < 2 \end{cases}$ *is continuous on* $\mathbb{R}$.

Exercises 5.1

1. Use definition to prove that f is continuous at a.

   1.1 $f(x) = x^2 + 1$ and $a = 1$.

   1.2 $f(x) = x^3$ and $a = -1$.

   1.3 $f(x) = \dfrac{1}{x}$ and $a = 1$.

   1.4 $f(x) = \dfrac{x}{x^2+1}$ and $a = 2$.

2. Determine whether the following functions are continuous at a.

   2.1 $f(x) = \begin{cases} 1-2x & \text{if } x \geq 1 \\ 2-3x & \text{if } x < 1 \end{cases}$ and $a = 1$

   2.2 $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 0 \\ \sqrt{1-x} & \text{if } x < 0 \end{cases}$ and $a = 0$

3. Use definition to prove that f is continuous at E.

   3.1 $f(x) = x^3$ and $E = \mathbb{R}$.

   3.2 $f(x) = \sqrt{1-x}$ and $E = (-\infty, 1)$.

   3.3 $f(x) = \dfrac{1}{x^2+1}$ and $E = \mathbb{R}$.

4. Use limit theorem to show that the following function are continuous on $[0, 1]$.

   4.1 $f(x) = 3x^2 + 1$

   4.2 $f(x) = \dfrac{1-x}{1+x}$

   4.3 $f(x) = \sqrt{2-x}$

   4.4 $f(x) = \dfrac{1}{x^2+x-6}$

5. Find a and b such that the function $f(x) = \begin{cases} ax+3 & \text{if } x \leq 1 \\ x+b & \text{if } 1 < x \leq 2 \\ 2ax-2 & \text{if } x > 2 \end{cases}$ is continuous on $\mathbb{R}$.

6. If $f : [a, b] \to \mathbb{R}$ is continuous, prove that $\sup\limits_{x\in[a,b]} |f(x)|$ is finite.

7. Show that there exist nowhere continuous functions f and g whose sum $f + g$ is continuous on $\mathbb{R}$. Show that the same is ture for product of functions.

8. Let

$$f(x) = \begin{cases} \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, 0)$ and $(0, \infty)$, discontinuous at 0, and neither $f(0^+)$ nor $f(0^-)$ exists.

8.1 Prove that f is continuous on $(-\infty, 0)$ and $(0, \infty)$ discontinuous at 0.

8.2 Suppose that $g : [0, \frac{2}{\pi}] \to \mathbb{R}$ is continuous on $(0, \frac{2}{\pi})$ and that there is a positive constant $C > 0$ such that

$$|g(x)| \leq C\sqrt{x} \text{ for all } x \in (0, \tfrac{2}{\pi}),$$

Prove that $f(x)g(x)$ is continuous on $[0, \frac{2}{\pi}]$.

9. Suppose that $a \in \mathbb{R}$, that I is an open interval containing a, that, $f, g : I \to \mathbb{R}$, and that f is continuous at a.

9.1 Prove that g is continuous at a if and only if $f + g$ is continuous at a.

9.2 Make and prove an analogous atstement for the product fg. Show by example that hypothesis about f added cannot be dropped.

10. Let $f : A \to \mathbb{R}$ be a continuous function. Suppose that $E \subseteq A$ and is open. Determine whether $\{f(x) : x \in E\}$ is open.

11. Let $f(x) = x^n$ where $n \in \mathbb{N}$. Prove that f is continuous on $\mathbb{R}$

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

12.1 Show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

12.2 Prove that $f(qx) = qf(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q}$.

12.3 Prove that f is continuous at 0 if and only if f is continuous on $\mathbb{R}$.

12.4 Prove that f is continuous at 0, then there is an $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

13. Assume that $\lim\limits_{n\to 0} \dfrac{\ln(x+1)}{x} = 1$ and $f(x) = e^x$ is continuous on $\mathbb{R}$. Show that $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

5.2 Intermediate Value Theorem

Definition 5.2.1 *Let E be a nonempty subsets of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be* ***bounded on E*** *if and only if there is an $M > 0$ such that*

$$|f(x)| \leq M \quad \text{for all } x \in E$$

Example 5.2.2 *Show that $f(x) = \dfrac{1}{x^2+1}$ is bounded on $\mathbb{R}$.*

Definition 5.2.3 *Let I be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Define*

$$\sup_{x\in I} f(x) := \sup\{f(x) : x \in I\} \quad \text{and} \quad \inf_{x\in I} f(x) := \inf\{f(x) : x \in I\}.$$

Example 5.2.4 *Let $f(x) = x^2$. Find a supremum and infimum of f on I.*

1. $I = [0, 1)$
2. $I = (-1, 1)$
3. $I = (-1, \infty)$

Theorem 5.2.5 (**Extreme Value Theorem (EVT)**) *If I is a closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover, if*

$$M = \sup_{x\in I} f(x) \quad \text{and} \quad m = \inf_{x\in I} f(x),$$

then there exist point $x_m, x_M \in I$ such that

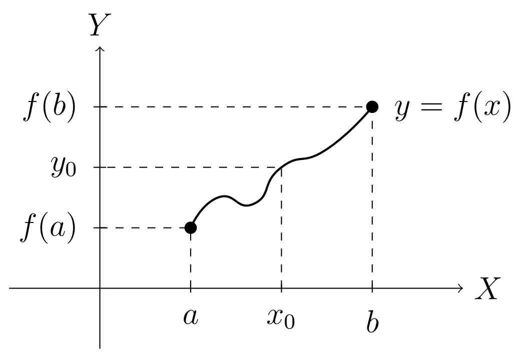
$$f(x_M) = M \quad \text{and} \quad f(x_m) = m.$$

Lemma 5.2.6 (**Sign-Preserving Property**) *Let $f : I \to \mathbb{R}$ where I is open. If f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$, then there are positive numbers ε and δ such that*

$$|x - x_0| < \delta \quad \textit{implies} \quad f(x) > \varepsilon.$$

Theorem 5.2.7 (**Intermediate Value Theorem (IVT)**) *Let $f : [a, b] \to \mathbb{R}$ be continuous. If y_0 lies between $f(a)$ and $f(b)$, then*

$$\text{there is an } x_0 \in (a, b) \text{ such that } f(x_0) = y_0.$$



Corollary 5.2.8 *Let $f : [a, b] \to \mathbb{R}$ be continuous.*

1. *If $f(a) > 0$ and $f(b) < 0$, then there is an $c \in (a, b)$ such that $f(c) = 0$.*

2. *If $f(a) < 0$ and $f(b) > 0$, then there is an $c \in (a, b)$ such that $f(c) = 0$.*

Example 5.2.9 *Show that there is a real number such that $x^2 = x + 1$.*

Example 5.2.10 *Show that there is a real number x such that $x^3 - x - 3 = 0$.*

Example 5.2.11 *Prove that*

$$\ln x = 3 - 2x$$

has at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ *of length 0.01 that contain a root.*

Exercises 5.2

For these exercise, assume that $\sin x$, $\cos x$ and e^x are continuous on $\mathbb{R}$ and $\ln x$ is continuous on $\mathbb{R}^+$.

1. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the given equation.

1.1 $x^3 + x = 3$	1.6 $e^x = x^2$
1.2 $x^3 + 2 = 2x$	1.7 $x \ln x = 1$
1.3 $x^4 + x^3 - 2 = 0$	1.8 $\sin x = e^x$
1.4 $x^5 + x + 1 = 0$	1.9 $\cos x = x^2$
1.5 $2^x = 2 - x$	1.10 $e^x = \cos x + 1$

2. Prove that the follwing equations have at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ of length ℓ that contain a root.

2.1 $x^3 + x = 1$	and $\ell = 0.001$	2.4 $\cos x = x$	and $\ell = 0.01$
2.2 $2^x = x^3$	and $\ell = 0.01$	2.5 $\sin x + x = 1$	and $\ell = 0.001$
2.3 $\ln x + x = 2$	and $\ell = 0.001$	2.6 $xe^x = \cos x$	and $\ell = 0.01$

3. Suppose that f is a real-value function of a real variable. If f is continuous at a with $f(a) < M$ for some $M \in \mathbb{R}$, prove that there is an open interval I containing a such that

$$f(x) < M \text{ for all } x \in I.$$

4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty,$$

prove that f has a minimum on $\mathbb{R}$; i.e., there is an $x_m \in \mathbb{R}$ such that

$$f(x_m) = \inf_{x\in\mathbb{R}} f(x) < \infty.$$

5.3 Uniform continuity

Definition 5.3.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be* ***uniformly continuous on E*** *if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.3.2 *Prove that $f(x) = x$ is uniformly continuous on $(0, 1)$.*

Example 5.3.3 *Prove that $f(x) = x^2$ is uniformly continuous on $(0, 1)$.*

Theorem 5.3.4 (**Uniform of continuity of Linear function**) *A Linear function is uniformly continuous on* $\mathbb{R}$.

Example 5.3.5 *Prove that* $f(x) = x^2$ *is not uniformly continuous on* $\mathbb{R}$.

Theorem 5.3.6 *Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.*

Theorem 5.3.7 *Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, then $f(x_n)$ is Cauchy.*

Exercises 5.3

1. Use Definition to prove that each of the following functions is uniformly continuous on $(0,1)$.

   1.1 $f(x) = x^3$
   1.2 $f(x) = x^2 - x$
   1.3 $f(x) = \dfrac{1}{x+1}$

2. Prove that each of the following functions is uniformly continuous on $(0,1)$.

   2.1 $f(x) = (x+1)^2$
   2.2 $f(x) = \dfrac{x^3-1}{x-1}$
   2.3 $f(x) = x\sin(\frac{1}{x})$
   2.4 $f(x)$ is any polynomial
   2.5 $f(x) = \dfrac{\sin x}{x}$
   2.6 $f(x) = x^2 \ln x$

3. Prove that $f(x) = \dfrac{1}{x^2+1}$ is uniformly continuous on $\mathbb{R}$.

4. Find all real α such that $x^\alpha \sin(\frac{1}{x})$ is uniformly continuous on the open interval $(0,1)$.

5. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous and there is an $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to \infty$. Prove that f is uniformly continuous on $[0,\infty)$.

6. Let I be a bounded interval. Prove that if $f : I \to \mathbb{R}$ is is uniformly continuous on I, then f is bounded on I.

7. Prove that (6) may be false if I is unbounded or if f is merely continuous.

8. Suppose that $\alpha \in \mathbb{R}$, E is nonempty subset of $\mathbb{R}$, and $f,g : E \to \mathbb{R}$ are uniformly continuous on E.

   8.1 Prove that $f+g$ and αf are uniformly continuous on E.

   8.2 Suppose that f,g are bounded on E. Prove that fg is uniformly continuous on E.

   8.3 Show that there exist functions f,g uniformly continuous on $\mathbb{R}$ such that fg is not uniformly continuous on $\mathbb{R}$.

9. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Chapter 6

Differentiability on $\mathbb{R}$

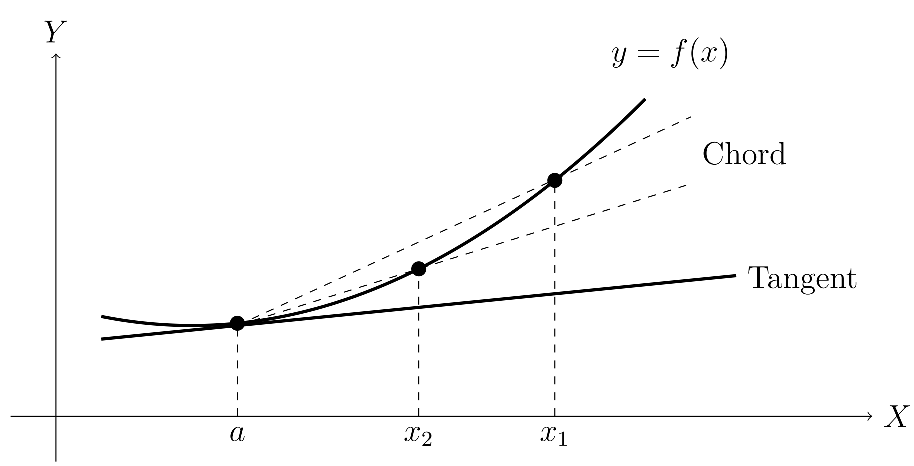
6.1 The Derivative

Definition 6.1.1 *A real function f is siad to be* ***differentiable*** *at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case $f'(a)$ is called the ***derivative*** *of f at a.*

You may recall that the graph of $y = f(x)$ has a **tangent line** at the point $(a, f(a))$ if and only if f has a derivative at a, in which case the slope of that tangent line is $f'(a)$. Suppose that f is differentiable at a. A **secant line** of the graph $y = f(x)$ is a line passing through at least two points on the graph, an a **chord** is a line segment that runs from one point on the graph to another.



Let $x = a + h$ and observe that the slope of the chord (chord function : $F(x)$) passing through the points $(x, f(x))$ and $(a, f(a))$ is given by

$$F(x) := \frac{f(x) - f(a)}{x - a}, \quad x \neq a.$$

Now, since $x = a + h$, $f'(a)$ becomes

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 6.1.2 *Let $f(x) = x^2$ where $x \in \mathbb{R}$. Find $f'(1)$*

Example 6.1.3 *Show that the function*

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at the origin.

Example 6.1.4 *Show that the function*

$$f(x) = \begin{cases} x\cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not differentiable at the origin.

Theorem 6.1.5 *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at a if and only if there is a function T of the form $T(x) := mx$ such that*

$$\lim_{h\to 0} \left| \frac{f(a+h) - f(a) - T(h)}{h} \right| = 0.$$

Theorem 6.1.6 *If f is differentiable at a, then f is continuous at a.*

Example 6.1.7 *Show that $f(x) = |x|$ is continuous at 0 but not differentiable there.*

DIFFERENTIABLE ON INTERVAL.

Definition 6.1.8 *Let I be an interval and $f : I \to \mathbb{R}$ be a function. f is said to be **differentiable on I** if and only if f is differentiable at a for every $a \in I$*

Example 6.1.9 *Show that the function $f(x) = x^2$ is differentiable on $\mathbb{R}$.*

Theorem 6.1.10 *Let $n \in \mathbb{N}$. If $f(x) = x^n$, then f is differentiable on $\mathbb{R}$ and*

$$f'(x) = nx^{n-1}.$$

Theorem 6.1.11 *Every constant function is differentiable on* $\mathbb{R}$ *and its value equals to zero.*

Example 6.1.12 *Show that* $f(x) = \sqrt{x}$ *is differentiable on* $(0, \infty)$ *and* $f'(x)$.

Example 6.1.13 *Show that* $f(x) = |x|$ *is differentiable on* $[0, 1]$ *and* $[-1, 0]$ *but not on* $[-1, 1]$.

Exercises 6.1

1. For each of the following real functions, use definition directly to prove that $f'(a)$ exists.

   1.1 $f(x) = x^3, \quad a \in \mathbb{R}$

   1.2 $f(x) = \dfrac{1}{x}, \quad a \neq 0$

   1.3 $f(x) = x^2 + x + 2, \quad a \in \mathbb{R}$

   1.4 $f(x) = \dfrac{1}{\sqrt{x}}, \quad a > 0$

2. Prove that $f(x) = x|x|$ is differentiable on $\mathbb{R}$.

3. Let I be an open interval that contains 0 and $f : I \to \mathbb{R}$. If there exists an $\alpha > 1$ such that

$$|f(x)| \leq |x|^\alpha \text{ for all } x \in I,$$

   prove that f is differentiable at 0. What happens when $\alpha = 1$?

4. Suppose that $f : (0, \infty) \to \mathbb{R}$ satisfies $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$ for all $x, y \in (0, \infty)$ and $f(1) = 0$.

   4.1 Prove that f is continuous on $(0, \infty)$ if and only if f is continuous at 1.

   4.2 Prove that f is differentiable on $(0, \infty)$ if and only if f is differentiable at 1.

   4.3 Prove that if f is differentiable at 1, then $f'(x) = \dfrac{f'(1)}{x}$ for all $x \in (0, \infty)$.

5. Suppose that $f_\alpha(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that $f_\alpha(x)$ is continuous at $x = 0$ when $\alpha > 0$ and differentiable at $x = 0$ when $\alpha > 1$. Graph these functions for $\alpha = 1$ and $\alpha = 2$ and give a geometric interpretation of your results.

6. Prove that if $f(x) = x^\alpha$ where $\alpha = \frac{1}{n}$ for somw $n \in \mathbb{N}$, then $y = f(x)$ is differentiable on $f'(x) = \alpha x^{\alpha - 1}$ for every $x \in (0, \infty)$.

7. Given $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$. Show that

   7.1 $(\sin x)' = \cos x$

   7.2 $(\cos x)' = -\sin x$

8. f is a constant function on I if and only if $f'(x) = 0$ for every $x \in I$.

6.2 Differentiability theorem

Theorem 6.2.1 (**Additive Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $f+g$ is differentiable at a. In fact,*

$$(f+g)'(a) = f'(a) + g'(a).$$

Theorem 6.2.2 (**Scalar Multiplicative Rule**) *Let f be a real function and $\alpha \in \mathbb{R}$. If f is differentiable at a, then αf is differentiable at a. In fact,*

$$(\alpha f)'(a) = \alpha f'(a).$$

Theorem 6.2.3 (**Product Rule**) *Let f and g be real functions. If f and g are differentiable at a, then fg is differentiable at a. In fact,*

$$(fg)'(a) = g(a)f'(a) + f(a)g'(a).$$

Theorem 6.2.4 (**Quotient Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $\frac{f}{g}$ is differentiable at a when $g(a) \neq 0$. In fact,*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}.$$

Example 6.2.5 *Let f and g be differentiable at 1 with $f(1) = 1, g(1) = 2$ and $f'(1) = 3, g'(1) = 4$. Evaluate the following derivatives.*

1. $(f+g)'(1)$
2. $(2f)'(1)$
3. $(fg)'(1)$
4. $\left(\dfrac{f}{g}\right)'(1)$

Theorem 6.2.6 (**Chain Rule**) *Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Example 6.2.7 *Let f and g be differentiable on $\mathbb{R}$ with $f(0) = 1, g(0) = -1$ and $f'(0) = 2$, $g'(0) = -2$,$f'(-1) = 3$, $g'(1) = 4$. Evaluate each of the following derivatives.*

1. $(f \circ g)'(0)$
2. $(g \circ f)'(0)$

Example 6.2.8 *Let $f(x) = \sqrt{x^2 + 1}$. Use the Chain Rule to show that $f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$.*

Exercises 6.2

1. For each of the following functions, find all x for which $f'(x)$ exists and find a formula for f'.

   1.1 $f(x) = \dfrac{x^3 - 2x^2 + 3x}{\sqrt{x}}$

   1.2 $f(x) = \dfrac{1}{x^2 + x - 1}$

   1.3 $f(x) = x|x|$

   1.4 $f(x) = |x^3 + 2x^2 - x - 2|$

2. Let f and g be differentiable at 2 and 3 with $f'(2) = a$, $f'(3) = b$, $g'(2) = c$ and $g'(3) = d$, If $f(2) = 1$, $f(3) = 2$,$g(2) = 3$ and $g(3) = 4$. Evaluate each of the following derivatives.

   2.1 $(fg)'(2)$

   2.2 $\left(\frac{f}{g}\right)'(3)$

   2.3 $(g \circ f)'(3)$

   2.4 $(f \circ g)'(2)$

3. If f, g and h is differentiable at a, prove that fgh is differentiable at a and
$$(fgh)'(a) = f'(a)g(a)h(a) + f(a)g'(a)h(a) + f(a)g(a)h'(a).$$

4. Let $f(x) = (x-1)(x-2)(x-3)\cdots(x-2565)$. Find $f'(2565)$

5. Prove that if $f(x) = x^{\frac{m}{n}}$ for some $n, m \in \mathbb{N}$, then $y = f(x)$ is differentiable and satisfies $ny^{n-1}y' = mx^{m-1}$ for every $x \in (0, \infty)$.

6. (**Power Rule**) Prove that $f(x) = x^q$ for some $q \in \mathbb{Q}$, then f is differentiable and $f'(x) = qx^{q-1}$ for every $x \in (0, \infty)$.

7. **(Reciprocal Rule)** Suppose that f is differentiable at a and $f(a) \neq 0$.

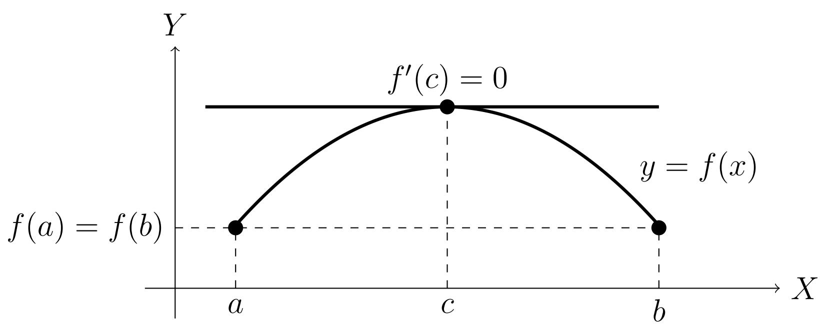
   7.1 Show that for h sufficiently small, $f(a+h) \neq 0$.

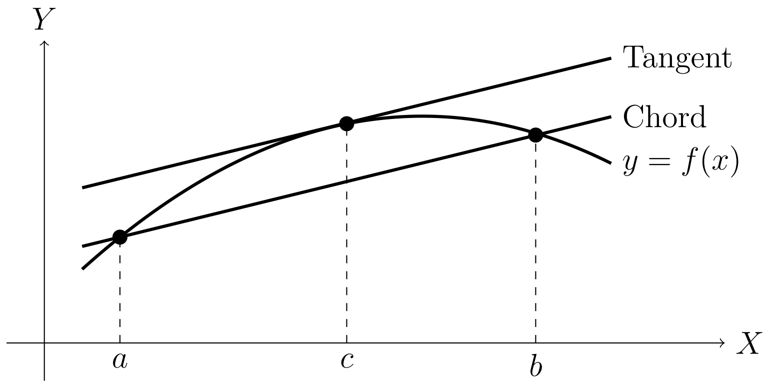
   7.2 Use Definition 6.1.1 directly, prove that $\dfrac{1}{f(x)}$ is differentiable at $x = a$ and
$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{f^2(a)}.$$

8. Suppose hat $n \in \mathbb{N}$ and f, g are real functions of a real variable whose nth derivatives $f^{(n)}, g^{(n)}$ exist at a point a. Prove Leibniz's generalization of the Product Rule:
$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a).$$

6.3 Mean Value Theorem

Lemma 6.3.1 (Rolle's Theorem) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.*





Theorem 6.3.2 (**Mean Value Theorem (MVT)**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is an $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Example 6.3.3 *Prove that*

$$\sin x \leq x \quad \textit{for all } x > 0.$$

Example 6.3.4 *Prove that*

$$1 + x \leq e^x \quad \textit{for all } x > 0.$$

Example 6.3.5 (**Bernoulli's Inequality**) *Let* $0 < \alpha \leq 1$ *and* $\delta \geq -1$. *Prove that*

$$(1+\delta)^\alpha \leq 1 + \alpha\delta.$$

Theorem 6.3.6 (**Generalized Mean Value Theorem**) *Suppose that* $a, b \in \mathbb{R}$ *with* $a \neq b$. *If* f *and* g *are continuous on* $[a, b]$ *and differentiable on* (a, b), *then there is an* $c \in (a, b)$ *such that*

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)].$$

Theorem 6.3.7 (**L'Hôspital's Rule**) *Let a be an extended real number and I be an open interval that either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \backslash \{a\}$, and $g(x) \neq 0 \neq g'(x)$ for all $x \in I \backslash \{a\}$. Suppose further that*

$$A := \lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

is either 0 or ∞. If

$$B := \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Given $(\ln x)' = \dfrac{1}{x}$ for $x > 0$ and $(e^x)' = e^x$ for all $x \in \mathbb{R}$.

Example 6.3.8 *Use L'Hôspital's Rule to prove that* $\displaystyle\lim_{x\to 0} \frac{x}{e^x - 1} = 1$.

Example 6.3.9 *Use L'Hôspital's Rule to find* $\displaystyle\lim_{x\to 0^+} x \ln x$.

Example 6.3.10 *Use L'Hôspital's Rule to find* $\displaystyle L = \lim_{x\to 1^-} (\ln x)^{1-x}$.

Exercises 6.3

1. Use the Mean Value Theorem to prove that each of the following inequalities.

   1.1 $\sqrt{2x+1} < 1 + x$ for all $x > 0$

   1.2 $\ln x \le x - 1$ for all $x > 1$

   1.3 $7(x-1) < e^x$ for all $x > 2$

   1.4 $\cos x - 1 \le x$ for all $x > 0$

   1.5 $\ln x + 1 \le \dfrac{x^2+1}{2}$ for all $x > 1$

   1.6 $\dfrac{x-1}{x} \le \ln x$ for all $x > 1$

   1.7 $\sqrt{x} \ge x$ for all $x \in [0, 1]$

   1.8 $\sqrt{x} \le x$ for all $x > 1$

   1.9 $\sin^2 x \le 2|x|$ for all $x \in \mathbb{R}$

   1.10 $\ln x \le \sqrt{x}$ for all $x > 1$

2. (**Bernoulli's Inequality**) Let $\alpha \ge 1$ and $\delta \ge -1$. Prove that

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

3. Use L'H$\hat{\boldsymbol{o}}$spital's Rule to evaluate the following limits.

   3.1 $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{x}$

   3.2 $\lim\limits_{x\to 0^+} \dfrac{\cos x - e^x}{\ln(1+x^2)}$

   3.3 $\lim\limits_{x\to 0} \left(\dfrac{x}{\sin x}\right)^{\frac{1}{x^2}}$

   3.4 $\lim\limits_{x\to 0^+} x^x$

   3.5 $\lim\limits_{x\to 1} \dfrac{\ln x}{\sin(\pi x)}$

   3.6 $\lim\limits_{x\to\infty} x\left(\arctan x - \dfrac{\pi}{2}\right)$

   3.7 $\lim\limits_{x\to 0^-} \left(1+e^{-x}\right)^x$

   3.8 $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}}$

   3.9 $\lim\limits_{x\to\infty} x(e^{\frac{1}{x}} - 1)$

4. Show that the derivative of

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   exists and continuous on $\mathbb{R}$ with $f'(0) = 0$.

5. Suppose that f is differentiable on $\mathbb{R}$.

   5.1 If $f'(x) = 0$ for all $x \in \mathbb{R}$, prove that $f(x) = f(0)$ for all $x \in \mathbb{R}$

   5.2 If $f(0) = 1$ and $|f'(x)| \le 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \le |x| + 1$ for all $x \in \mathbb{R}$

   5.3 If $'(x) \ge 0$ for all $x \in \mathbb{R}$, prove that $a < b$ imply that $f(a) < f(b)$

6. Let f be differentiable on a nonempty, open interval (a, b) with f' bounded on (a, b). Prove that f is uniformly continuous on (a, b).

7. Let f be differentiable on (a, b), continuous on $[a, b]$, with $f(a) = f(b) = 0$. Prove that if $f'(c) > 0$ for some $c \in (a, b)$, then there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) > 0 > f'(x_2)$.

8. Let f be twice differentiable on (a, b) and let there be points $x_1 < x_2 < x_3$ in (a, b) such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there is a point $c \in (a, b)$ such that $f''(c) > 0$.

9. Let f be differentiable on $(0, \infty)$. If $L = \lim_{x\to\infty} f'(x)$ and $\lim_{n\to\infty} f(n)$ both exist and are finite, prove that $L = 0$.

10. Prove L'Hôspital's Rule for the case $B = \pm\infty$ by first proving that

$$\frac{g(x)}{f(x)} \to 0 \text{ when } \frac{f(x)}{g(x)} \to \pm\infty, \text{ as } x \to a.$$

11. Prove that the sequence $\left(1 + \frac{1}{n}\right)^n$ is increasing, as $n \to \infty$, and its limit e satisfies $2 < e \leq 3$ and $\ln e = 1$.

6.4 Monotone function

Definition 6.4.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*

1. *f is said to be **increasing** on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \leq f(x_2).$$

*f is said to be **strictly increasing** on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) < f(x_2).$$

2. *f is said to be **decreasing** on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \geq f(x_2).$$

*f is said to be **strictly decreasing** on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) > f(x_2).$$

3. *f is said to be **monotone** on E if and only if f is either decreasing or increasing on E.*
   *f is said to be **strictly monotone** on E if and only if f is either strictly decreasing or strictly increasing on E.*

Example 6.4.2 *Show that $f(x) = x^2$ is strictly monotone on $[0, 1]$ and on $[-1, 0]$ but not monotone on $[-1, 1]$.*

Theorem 6.4.3 *Let* $f : I \to \mathbb{R}$ *and* $(a, b) \subseteq I$*. Then*

1. f *is increasing on* (a, b) *if* $f'(x) > 0$ *for all* $x \in (a, b)$
2. f *is decreasing on* (a, b) *if* $f'(x) < 0$ *for all* $x \in (a, b)$
3. *If* $f'(x) = 0$ *for all* $x \in (a, b)$*, then* f *is constant on* $[a, b]$*.*

Example 6.4.4 *Find each intervals of* $f(x) = x^2 - 4x + 3$ *that increasing and decreasing.*

Theorem 6.4.5 *If f is 1-1 and continuous on an interval I, then f is strictly monotone on I and f^{-1} is continuous and strictly monotone on $f(I) := \{f(x) : x \in I\}$.*

Theorem 6.4.6 (**Inverse Function Theorem (IFT)**) *Let f be 1-1 and continuous on an open interval I. If $a \in f(I)$ and if $f'(f^{-1}(a))$ exists and is nonzero, then f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Example 6.4.7 *Use the IVT to find derivative of $f(x) = \arcsin x$*

Example 6.4.8 *Let* $f(x) = x + e^x$ *where* $x \in \mathbb{R}$.

1. *Show that* f *is 1-1 on* $x \in \mathbb{R}$.
2. *Use the result from 1 and the IFT to explain that* f^{-1} *differentiable on* $\mathbb{R}$.
3. *Compute* $(f^{-1})'(2 + \ln 2)$.

Exercises 6.4

1. Find each intervals of the following functions that increasing and decreasing.

   1.1 $f(x) = 2x - x^2$

   1.2 $f(x) = x^3 - x^2 - x + 3$

   1.3 $f(x) = (x-1)^3(x-2)^4$

   1.4 $g(x) = xe^x$

   1.5 $g(x) = e^x - x$

   1.6 $g(x) = x^2e^{x^2}$

2. Find all $a \in \mathbb{R}$ such that $x^3 + ax^2 + 3x + 15$ is strictly increasing near $x = 1$.

3. Find all $a \in \mathbb{R}$ such that $ax^2 + 3x + 5$ is strictly increasing on the interval $(1, 2)$.

4. Find where $f(x) = 2|x-1| + 5\sqrt{x^2+9}$ is strictly increasing and where $f(x)$ is strictly decreasing.

5. Let f and g be 1-1 and continuous on $\mathbb{R}$. If $f(0) = 2$, $g(1) = 2$, $f'(0) = \pi$, and $g'(1) = e$, compute the following derivatives.

   5.1 $(f^{-1})'(2)$

   5.2 $(g^{-1})'(2)$

   5.3 $(f^{-1} \cdot g^{-1})'(2)$

6. Let $f(x) = x^2e^{x^2}$, $x \in \mathbb{R}$.

   6.1 Show that f^{-1} exists and its differentiable on $(0, \infty)$.

   6.2 Compute $(f^{-1})'(e)$

7. Let $f(x) = x + e^{2x}$ where $x \in \mathbb{R}$.

   7.1 Show that f is 1-1 on $x \in \mathbb{R}$.

   7.2 Use the result from 7.1 and the IFT to explain that f differentiable on $\mathbb{R}$.

   7.3 Compute $(f^{-1})'(4 + \ln 2)$.

8. Use the Inverse Function Theorem, prove that

   8.1 $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$ where $x \in (-1, 1)$

   8.2 $(\arctan x)' = \dfrac{1}{1+x^2}$ where $x \in (-\infty, \infty)$

8.3 $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ where $x \in (0, \infty)$

9. Use the IFT to find derivative of invrese function $f(x) = e^x - e^{-x}$ where $x \in \mathbb{R}$.

10. Suppose that f' exists and continuous on a nonempty, open interval (a, b) with $f'(x) \neq 0$ for all $x \in (a, b)$.

    10.1 Prove that f is 1-1 on (a, b) and takes (a, b) onto some open interval (c, d)

    10.2 Show that $(f^{-1})'$ exists and continuous on (c, d)

    10.3 Use the function $f(x) = x^3$, show that 7.2 is false if the assumption $f'(x) \neq 0$ fails to hold for some $x \in (c, d)$

11. Let $[a, b]$ be a closed, bounded interval. Find all functions f that satisfy the following conditions for some fixed $\alpha > 0$: f is continuous and 1-1 on $[a, b]$,

$$f'(x) \neq 0 \text{ and } f'(x) = \alpha(f^{-1})'(f(x)) \text{ for all } x \in (a, b).$$

12. Let f be differentiable at every point in a closed, bounded interval $[a, b]$. Prove that if f' is increasing on (a, b), then f' is continuous on (a, b).

13. Suppose that f is increasing on $[a, b]$. Prove that

    13.1 if $x_0 \in [a, b)$, then $f(x_0^+)$ exists and $f(x_0) \leq f(x_0^+)$,

    13.2 if $x_0 \in (a, b]$, then $f(x_0^-)$ exists and $f(x_0^-) \leq f(x_0)$.

Chapter 7

Integrability on $\mathbb{R}$

7.1 Riemann integral

PARTITION.

Definition 7.1.1 *Let $a, b \in \mathbb{R}$ with $a < b$.*

1. *A **partition** of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, , ..., x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

2. *The **norm** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is the number*

$$\|P\| = \max_{1 \leq j \leq n} |x_j - x_{j-1}|.$$

3. *A **refinement** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is a partition Q of $[a, b]$ that satisfies $Q \supseteq P$. In this case we say that Q is **finer** than P or Q is a **refinement** of P.*

Example 7.1.2 *Give example of partition and refinement of the interval $[0, 1]$.*

Partitions	Norms of Partition
$P = \{0, 0.5, 1\}$	
$Q = \{0, 0.25, 0.5, 0.75, 1\}$	
$R = \{0, 0.2, 0.3, , 0.5, 0.6, 0.8, 1\}$	

Example 7.1.3 *Prove that for each* $n \in \mathbb{N}$,

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

is a partition of the interval $[0, 1]$ *and find a norm of* P_n.

Example 7.1.4 (**Dyadic Partition**) *Let* $n \in \mathbb{N}$ *and define*

$$P_n = \left\{ \frac{j}{2^n} : j = 0, 1, ..., 2^n \right\}.$$

1. *Prove that* P_n *is a partition of the interval* $[0, 1]$.

2. *Prove that* P_m *is finer than* P_n *when* $m > n$.

3. *Find a norm of* P_n.

UPPER AND LOWER RIEMANN SUM.

Definition 7.1.5 *Let $a, b \in \mathbb{R}$ with $a < b$, let $P = \{x_0, x_1, , ..., x_n\}$ be a partition of the interval $[a, b]$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded.*

1. *The* ***upper Riemann sum*** *of f over P is the number*

$$U(f, P) := \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1})$$

   where

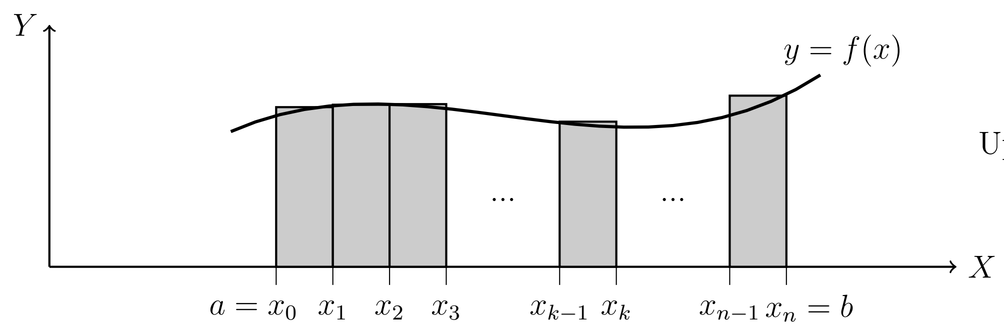
$$M_j(f) := \sup_{x \in [x_{j-1}, x_j]} f(x).$$

2. *The* ***lower Riemann sum*** *of f over P is the number*

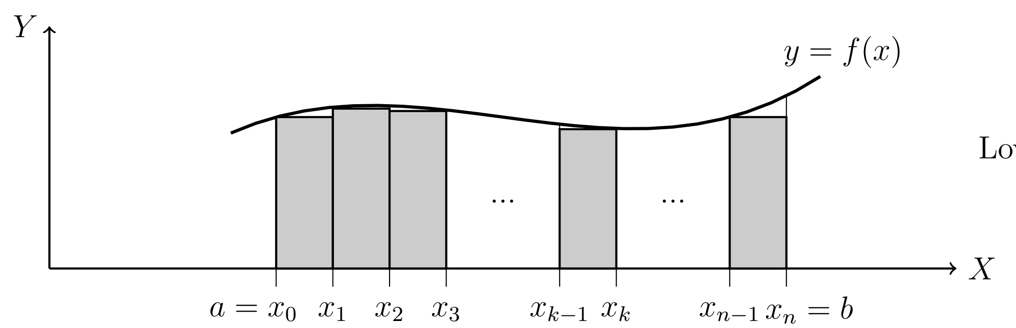
$$L(f, P) := \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1})$$

   where

$$m_j(f) := \inf_{x \in [x_{j-1}, x_j]} f(x).$$



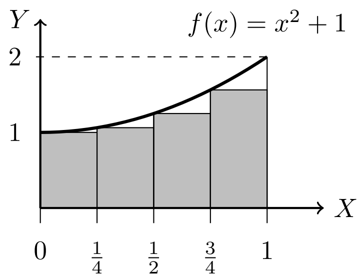
Upper Riemann Sum

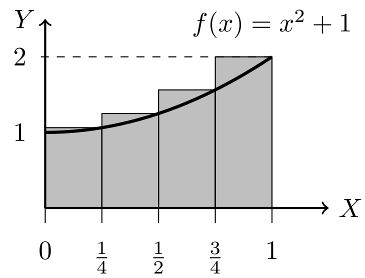


Lower Riemann Sum

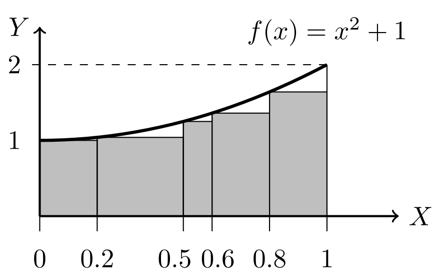
Example 7.1.6 *Let $f(x) = x^2 + 1$ where $x \in [0, 1]$. Find $L(f, P)$ and $U(f, P)$*

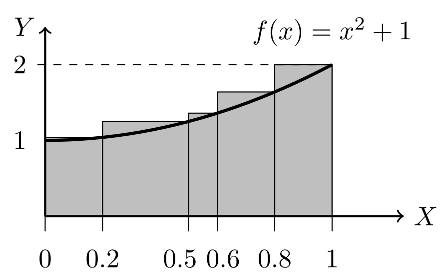
1. $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$





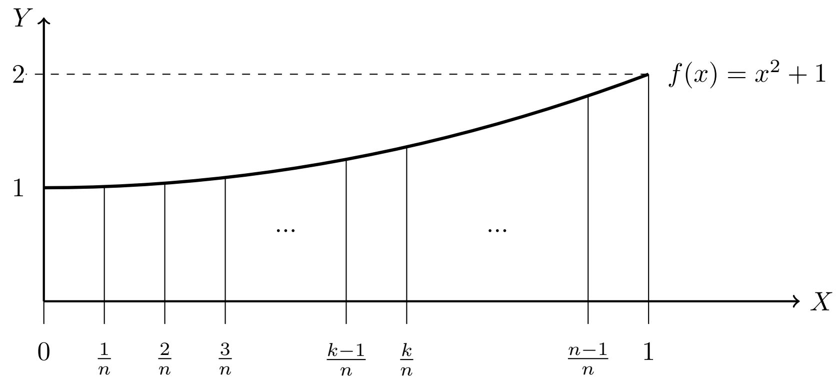
2. $P = \{0, 0.2, 0.5, 0.6, 0.8, 1\}$





Example 7.1.7 *Let* $f(x) = x^2 + 1$ *where* $x \in [0, 1]$. *Find* $L(P_n, f)$ *and* $U(P_n, f)$ *for* $n \in \mathbb{N}$ *if*

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}.$$



Theorem 7.1.8 *$L(f,P) \leq U(f,P)$ for all partition P and all bounded function f.*

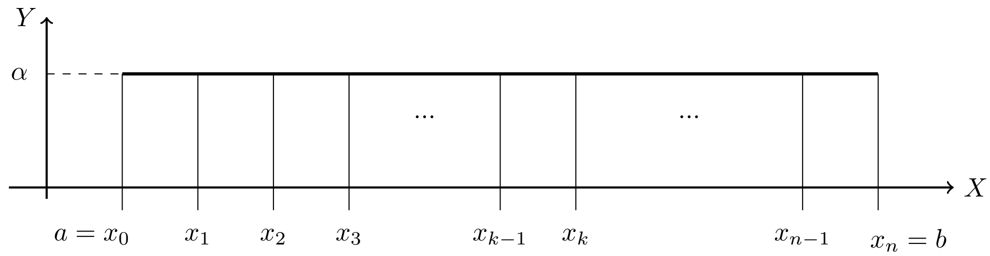
Theorem 7.1.9 (Sum Telescopes) *If $g : \mathbb{N} \to \mathbb{R}$, then*

$$\sum_{k=m}^{n} [g(k+1) - g(k)] = g(n+1) - g(m)$$

for all $n \geq m$ in $\mathbb{N}$.

Theorem 7.1.10 *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$U(f, P) = L(f, P) = \alpha(b - a)$$



Theorem 7.1.11 *If P is any partition of $[a, b]$ and Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Corollary 7.1.12 *If P and Q are any partitions of $[a, b]$, then*

$$L(f, P) \leq U(f, Q).$$

RIEMANN INTEGRABLE.

Definition 7.1.13 *Let $a, b \in \mathbb{R}$ with $a < b$.*

A function $f : [a, b] \to \mathbb{R}$ is said to be ***Riemann integrable*** *or* ***integrable*** *on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 7.1.14 *Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.*

Example 7.1.15 *Prove that the function*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

is integrable on $[0, 1]$.

Example 7.1.16 (Dirichlet function) *Prove that the function*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is NOT Riemann integrable on $[0, 1]$.

UPPER AND LOWER INTEGRABLE.

Definition 7.1.17 *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded.*

1. *The* ***upper integral*** *of f on $[a, b]$ is the number*

$$(U) \int_a^b f(x)\,dx := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

2. *The* ***lower integral*** *of f on $[a, b]$ is the number*

$$(L) \int_a^b f(x)\,dx := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

3. *If the upper and lower integrals of f on $[a, b]$ are equal, we define the* ***integral*** *of f on $[a, b]$ to be the common value*

$$\int_a^b f(x)\,dx := (U) \int_a^b f(x)\,dx = (L) \int_a^b f(x)\,dx.$$

Example 7.1.18 *Let $f(x) = \alpha$ where $x \in [a, b]$. Show that*

$$(U) \int_a^b f(x)\,dx = (L) \int_a^b f(x)\,dx = \alpha(b - a).$$

Example 7.1.19 *The Dirichlet function is defined*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Find the upper integral and lower integral of the Dirichlet function on $[0, 1]$.

Theorem 7.1.20 *If $f : [a,b] \to \mathbb{R}$ is bounded, then its upper and lower integrals exist and are finite, and satisfy*

$$(L)\int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

Theorem 7.1.21 *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if*

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx.$$

Theorem 7.1.22 *For a constant α,*

$$\int_a^b \alpha\, dx = \alpha(b-a).$$

Example 7.1.23 *Let $f : [0,2] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

Show that f is integrable and find $\int_0^2 f(x)dx$.

Example 7.1.24 *Let* $f : [0,1] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 2 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Show that f *is integrable and find* $\displaystyle\int_0^1 f(x)dx$.

Exercises 7.1

1. For each of the following, compute $U(f,P)$, $L(f,P)$, and $\int_0^1 f(x)\,dx$, where

$$P = \left\{0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1\right\}.$$

Find out whether the lower sum or the upper sum is better approximation to the integral. Graph f and explain why this is so.

1.1 $f(x) = 1 - x^2$ 1.2 $f(x) = 2x^2 + 1$ 1.3 $f(x) = x^2 - x$

2. Let $P_n = \left\{\frac{j}{n} : n = 0, 1, ..., n\right\}$ for each $n \in \mathbb{N}$. Prove that a bounded function f is integrable on $[0,1]$ if

$$I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

in which case $\int_0^1 f(x)\,dx$ equals I_0.

3. For each of the following functions, use P_n in 2. to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)\,dx$.

3.1 $f(x) = x$

3.2 $f(x) = x^2$

3.3 $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$

4. Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that the function $f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if otherwise} \end{cases}$ is integrable on $[0,1]$. What is the value of $\int_0^1 f(x)\,dx$?

5. Suppose that f is continuous on an interval $[a,b]$. Show that $\int_a^c f(x)\,dx = 0$ for all $c \in [a,b]$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

6. Let f be bounded on a nondegenerate interval $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that

$$P \supseteq P_\varepsilon \quad \text{imples} \quad |U(f,P) - L(f,P)| < \varepsilon.$$

7.2 Riemann sums

Definition 7.2.1 *Let $f : [a, b] \to \mathbb{R}$.*

1. *A* ***Riemann sum*** *of f with respect to a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ is a sum of the form*

$$\sum_{j=1}^{n} f(t_j)\Delta x_j,$$

   where the choice of $t_j \in [x_{j-1}, x_j]$ is arbitrary.

2. *The Riemann sums of f are* ***converge*** *to $I(f)$ as $\|P\| \to 0$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that*

$$P = \{x_0, x_1, ..., x_n\} \supseteq P_\varepsilon \quad \textit{implies} \quad \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \varepsilon$$

   for all choice of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$. In this case we shall use the notation

$$I(f) = \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j.$$

Example 7.2.2 *Let* $f(x) = x^2$ *where* $x \in [0,1]$ *and*

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. *Show that if* $f(t_i)$ *is choosen by the right end point and left end point in each subinterval, then two* $I(f)$, *depend on two methods, are NOT different.*

Theorem 7.2.3 *Let* $a, b \in \mathbb{R}$ *with* $a < b$, *and suppose that* $f : [a, b] \to \mathbb{R}$ *is bounded. Then* f *is Riemann integrable on* $[a, b]$ *if and only if*

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j$$

exists, in which case

$$I(f) = \int_a^b f(x)\, dx.$$

Theorem 7.2.4 (**Linear Property**) *If f, g are integrable on $[a, b]$ and $\alpha \in \mathbb{R}$, then $f + g$ and αf are integrable on $[a, b]$. In fact,*

1. $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

Theorem 7.2.5 *If f is integrable on $[a,b]$, then f is integrable on each subinterval $[c,d]$ of $[a,b]$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for all $c \in (a,b)$.

By Theorem 7.2.5, we obtain

$$\int_a^b f(x)\,dx = \int_a^a f(x)\,dx + \int_a^b f(x)\,dx$$

Thus,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 7.2.6 *Using the connection between integrals are area, evaluate* $\int_0^5 |x-2|\,dx$.

Example 7.2.7 *Using the connection between integrals are area, evaluate* $\int_0^2 \sqrt{4-x^2}\,dx$.

Theorem 7.2.8 (Comparison Theorem) *If f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In particular, if $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Theorem 7.2.9 *If f is Riemann integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

Exercises 7.2

1. Using the connection between integrals are area, evaluate each of the following integrals.

   1.1 $\int_0^1 |x - 0.5|\, dx$

   1.2 $\int_0^a \sqrt{a^2 - x^2}\, dx, \quad a > 0$

   1.3 $\int_{-2}^2 (|x+1| + |x|)\, dx$

   1.4 $\int_a^b (3x+1)\, dx, \quad a < b$

2. Prove that if f is integrable on $[0,1]$ and $\beta > 0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{\frac{1}{n^\beta}} f(x)\, dx = 0 \qquad \text{for all } \alpha < \beta.$$

3. If f, g are integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, prove that

$$\left|\int_a^b (f(x) + g(x))\, dx\right| \le \int_a^b |f(x)|\, dx + \int_a^b |g(x)|\, dx.$$

4. Suppose that $g_n \ge 0$ is a sequence of integrable function that satisfies $\lim_{n\to\infty} \int_a^b g_n(x)\, dx = 0$.

   Show that if $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, then $\lim_{n\to\infty} \int_a^b f(x) g_n(x)\, dx = 0$.

5. Prove that if f is integrable on $[0,1]$, then $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0$.

6. Prove that if f is integrable on $[0,1]$, then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} f(x)\, dx.$$

7. Let f be continuous on a closed, nondegenerate interval $[a,b]$ and set $M = \sup_{x\in[a,b]} |f(x)|$.

   7.1 Prove that if $M > 0$ and $p > 0$, then for every $\varepsilon > 0$ there is a nondegenerate on interval $I \subset [a,b]$ such that

$$(M - \varepsilon)^p |I| \le \int_a^b |f(x)|^p\, dx \le M^p (b-a).$$

   7.2 Prove that $\lim_{p\to\infty} \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} = M.$

7.3 Fundamental Theorem of Calculus

Define a set $C^1[a,b] = \{f : [a,b] \to \mathbb{R} : f$ is differentiable and f' are continuous $\}$ and $f'(x) = \dfrac{df}{dx}$.

Theorem 7.3.1 (**Fundamental Theorem of Calculus**) *Suppose that* $f : [a,b] \to \mathbb{R}$.

1. *If* f *is continuous on* $[a,b]$ *and* $F(x) = \int_a^x f(t)\,dt$, *then* $F \in C^1[a,b]$ *and*

$$\frac{d}{dx}\int_a^x f(t)\,dt := F'(x) = f(x)$$

   for each $x \in [a,b]$.

2. *If* f *is differentiable on* $[a,b]$ *and* f' *is integrable on* $[a,b]$, *then*

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

   for each $x \in [a,b]$.

Example 7.3.2 *Assume that f is differentiable on $(0,1)$ and integrable on $[0,1]$. Show that*

$$\int_0^1 xf'(x) + f(x)\,dx = f(1).$$

Theorem 7.3.3 *Let $\alpha \neq -1$. Then*

$$\int_a^b x^\alpha\,dx = f(b) - f(a) \quad \text{where } f(x) = \frac{x^{\alpha+1}}{\alpha+1}.$$

Example 7.3.4 *Find integral* $\int_0^1 x^2\,dx$.

Theorem 7.3.5 *Suppose that* $f, u : [a, b] \to \mathbb{R}$*. If* f *is continuous on* $[a, b]$ *and* $F(x) = \int_a^{u(x)} f(t)\, dt$*, and* $F \in C^1[a, b]$ *and*

$$F'(x) = \frac{d}{dx} \int_a^{u(x)} f(t)\, dt = f(u(x)) \cdot u'(x)$$

for each $x \in [a, b]$*.*

Example 7.3.6 *Let* $F(x) = \int_0^{\sin x} e^{t^2}\, dt$*. Find* $F(0)$ *and* $F'(0)$*.*

INTEGRATION BY PART.

Theorem 7.3.7 (**Integration by Part**) *Suppose that f, g are differentiable on $[a, b]$ with f', g' integrable on $[a, b]$, Then*

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Example 7.3.8 *Use the Integration by Part to find integrals.*

1. $\int_0^{\frac{\pi}{2}} x \sin x \, dx$
2. $\int_1^2 \ln x \, dx$

Example 7.3.9 *Let* $f(x) = \displaystyle\int_0^{x^3} e^{t^2} dt$*. Use integration by part to show that*

$$6\int_0^1 x^2 f(x)dx - 2\int_0^1 e^{x^2} dx = 1 - e.$$

CHANGE OF VARIABLES.

Theorem 7.3.10 (**Change of Variables**) *Let ϕ be continuously differentiable on a closed interval $[a, b]$. If f is continuous on $\phi([a, b])$, or if ϕ is strictly incresing on $[a, b]$ and f is integrable on $[\phi(a), \phi(b)]$, then*

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\phi'(x)\,dx.$$

Example 7.3.11 *Find* $\displaystyle\int_0^3 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

Example 7.3.12 *Evaluate*

$$\int_{-1}^{1} x f(x^2)\, dx$$

for any f *is continuous on* $[0, 1]$.

Example 7.3.13 *Let* $f : [-a, a] \to \mathbb{R}$ *where* $a > 0$. *Suppose* $f(-x) = -f(x)$ *for all* $x \in [-a, a]$. *Show that*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Exercises 7.3

1. Compute each of the following integrals.

   1.1 $\displaystyle\int_{-3}^{3} |x^2 + x - 2|\, dx$

   1.2 $\displaystyle\int_{1}^{4} \frac{\sqrt{x}-1}{\sqrt{x}}\, dx$

   1.3 $\displaystyle\int_{0}^{1} (3x+1)^{99}\, dx$

   1.4 $\displaystyle\int_{1}^{e} x \ln x\, dx$

   1.5 $\displaystyle\int_{0}^{\frac{\pi}{2}} e^x \sin x\, dx$

   1.6 $\displaystyle\int_{0}^{1} \sqrt{\frac{4x^2-4x+1}{x^2-x+3}}\, dx$

2. Use First Mean Value Theorem for Integrals to prove the followingversion of the Mean Value Theorem for Derivatives. If $f \in C^1[a,b]$, then there is an $x_0 \in [a,b]$ such that

$$f(b) - f(a) = (b-a)f'(x_0).$$

3. If $f : [0,\infty) \to \mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dx}\int_{0}^{x^2} f(t)\, dt$.

4. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dt}\int_{\cos t}^{t} g(x)\, dx$.

5. Let g be differentiable and integrable on $\mathbb{R}$. Define $f(x) = \displaystyle\int_{1}^{x^2} g(t)\cdot\sqrt{t}\, dt$.

   Show that $\displaystyle\int_{0}^{1} xg(x) + f(x)\, dx = 0$.

6. If $f(x) = \displaystyle\int_{0}^{x^2} \sec^2(t^2) dt$. show that $\displaystyle 2\int_{0}^{1} \sec^2(x^2)\, dx - 4\int_{0}^{1} xf(x)\, dx = \tan 1$.

7. Suppose that g is integrable and nonnegative on $[1,3]$ with $\displaystyle\int_{1}^{3} g(x)\, dt = 1$. Prove that

$$\frac{1}{\pi}\int_{1}^{9} g(\sqrt{x})\, dx < 2.$$

8. Suppose that h is integrable and nonnegative on $[1,11]$ with $\displaystyle\int_{1}^{11} h(x)\, dt = 3$. Prove that

$$\int_{0}^{2} h(1+3x+3x^2-x^3)\, dx \le 1.$$

9. If f is continuous on $[a,b]$ and there exist numbers $\alpha \neq \beta$ such that

$$\alpha\int_{a}^{c} f(x)\, dx + \beta\int_{c}^{b} f(x)\, dx = 0$$

   holds for all $c \in (a,b)$, prove that $f(x) = 0$ for all $x \in [a,b]$.

Chapter 8

Infinite Series of Real Numbers

8.1 Introduction

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of numbers. We shall call an expression of the form

$$\sum_{k=1}^{\infty} a_k$$

an **infinite series** with terms a_k.

Definition 8.1.1 *Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series whose terms a_k belong to $\mathbb{R}$.*

1. *The **partial sums** of S of order n are the numbers defined, for each $n \in \mathbb{N}$, by*

$$s_n := \sum_{k=1}^{n} a_k.$$

2. *S is said to **converge** if and only if its sequence of partial sums $\{s_n\}$ to some $s \in \mathbb{R}$ as $n \to \infty$; i.e., for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad |s_n - s| < \varepsilon.$$

In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call s the sum, or value, of the series $\sum_{k=1}^{\infty} a_k$.

3. *S is said to **diverge** if and only if its sequence of partial sums $\{s_n\}$ does not converge.*

Example 8.1.2 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]=1.$

Example 8.1.3 *Prove that* $\displaystyle\sum_{k=1}^{\infty}(-1)^k$ *diverges.*

Theorem 8.1.4 (Harmonic Series) *Prove that the sequence* $\frac{1}{k}$ *converges but the series*

$$\sum_{k=1}^{\infty}\frac{1}{k} \quad \textit{diverges.}$$

Theorem 8.1.5 (Divergence Test) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *be a sequence of real numbers.*

$$\text{If } a_k \text{ does not converge to zero, then the series } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Example 8.1.6 *Show that the series* $\displaystyle\sum_{k=1}^{\infty} \frac{n}{n+1}$ *diverges.*

Theorem 8.1.7 (**Telescopic Seires**) *If* $\{a_k\}$ *is a convergent real sequence, then*

$$\sum_{k=m}^{\infty}(a_k - a_{k+1}) = a_m - \lim_{k\to\infty} a_k.$$

Example 8.1.8 *Evaluate the series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}$.

Example 8.1.9 *Determine whether* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+1}+\sqrt{k}}$ *converges or not.*

Theorem 8.1.10 (Geometric Seires) *The series* $\sum_{k=1}^{\infty} x^k$ *converges if and only if* $|x| < 1$, *in which case*

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}.$$

Example 8.1.11 *Determine whether the following series converges or diverges.*

1. $\sum_{k=1}^{\infty} 2^{-k}$

2. $\sum_{k=1}^{\infty} (\sqrt{2}-1)^{-k}$

Theorem 8.1.12 *Let* $\{a_k\}$ *and* $\{b_k\}$ *be a real sequences. If* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are convergent series, then*

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \quad \text{and} \quad \sum_{k=1}^{\infty}(\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$

for any $\alpha \in \mathbb{R}$.

Theorem 8.1.13 *If* $\sum_{k=1}^{\infty} a_k$ *converges and* $\sum_{k=1}^{\infty} b_k$ *diverges, then*

$$\sum_{k=1}^{\infty}(a_k + b_k) \text{ diverges.}$$

Example 8.1.14 *Evaluate* $\sum_{k=1}^{\infty} \frac{1+2^{k+1}}{3^k}$.

Example 8.1.15 *Evaluate* $\sum_{k=1}^{\infty} \frac{k}{2^k}$.

Example 8.1.16 *Evaluate* $\sum_{k=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{5^k}{2^k} \right)$.

Example 8.1.17 *Let* π *be a Pi constant. Show that*

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right]$$

converges and find its value.

Example 8.1.18 *Evaluate the series* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$.

Exercises 8.1

1. Show that
$$\sum_{k=n}^{\infty} x^k = \frac{x^n}{1-x}$$
for $|x| < 1$ and $n = 0, 1, 2, ...$

2. Prove that each of the following series converges and find its value.

   2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}}$

   2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi^k}$

   2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}+4}{5^k}$

   2.4 $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{7^{k-1}}$

   2.5 $\displaystyle\sum_{k=0}^{\infty} 2^k e^{-k}$

   2.6 $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{2^k}$

3. Represent each of the following series as a telescopic series and find its value.

   3.1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$

   3.2 $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right)$

   3.3 $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{\frac{\pi}{4}}\left(1-\left(\frac{\pi}{4}\right)^{j_k}\right)$, where $j_k = -\dfrac{1}{k(k+1)}$ for $k \in \mathbb{N}$

4. Find all $x \in \mathbb{R}$ for which
$$\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1})$$
converges. For each such x, find the value of this series.

5. Prove that each of the following series diverges.

   5.1 $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k^2}$

   5.2 $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^k$

   5.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

6. Prove that if $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

7. Let $\{b_k\}$ be a real sequence and $b \in \mathbb{R}$.

7.1 Suppose that there is an $N \in \mathbb{N}$ such that $|b - b_k| \leq M$ for all $k \geq N$. Prove that

$$\left| nb - \sum_{k=1}^{n} b_k \right| \leq \sum_{k=1}^{N} |b_k - b| + M(n - N)$$

for all $n > N$.

7.2 Prove that if $b_k \to b$ as $k \to \infty$, then

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \to b \quad \text{as} \quad n \to \infty.$$

7.3 Show that converse of 7.2 is false.

8. A series $\sum_{k=0}^{\infty} a_k$ is said to be **Cesàro summable** to $L \in \mathbb{R}$ if and only if

$$\sigma_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k$$

converges to L as $n \to \infty$.

8.1 Let $s_n = \sum_{k=0}^{\infty} a_k$. Prove that $\sigma_n = \dfrac{s_1 + s_2 + \cdots + s_n}{n}$ for each $n \in \mathbb{N}$.

8.2 Prove that if $a_k \in \mathbb{R}$ and $\sum_{k=0}^{\infty} a_k = L$ converges, then c is Cesàro summable to L.

8.3 Prove that $\sum_{k=0}^{\infty} (-1)^k$ is Cesàro summable to $\frac{1}{2}$; hence the converge of 8.2 is false.

8.4 **TAUBER.** Prove that if $a_k \geq 0$ for $k \in \mathbb{N}$ and $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L, then $\sum_{k=0}^{\infty} a_k = L$.

9. Suppose that $\{a_k\}$ is a decreasing sequence of real numbers. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $ka_k \to 0$ as $k \to \infty$.

10. Suppose that $a_k \geq 0$ for k large and $\sum_{k=0}^{\infty} \dfrac{a_k}{k}$ converges. Prove that $\lim_{j \to \infty} \sum_{k=1}^{\infty} \dfrac{a_k}{j + k} = 0$.

11. If and $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges.

8.2 Series with nonnegative terms

INTEGRAL TEST.

Theorem 8.2.1 (**Integral Test**) *Suppose that* $f : [1, \infty) \to \mathbb{R}$ *is positive and decreasing on* $[1, \infty)$. *Then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if*

$$\lim_{n \to \infty} \int_1^n f(x)\, dx < \infty.$$

Example 8.2.2 *Use the Integral Test to prove that* $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

Example 8.2.3 *Show that* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges.*

Example 8.2.4 *Show that* $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *converges.*

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p-SERIES TEST.

Theorem 8.2.5 (p-Series Test) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Example 8.2.6 *Find* $p \in \mathbb{R}$ *such that* $\displaystyle\sum_{k=1}^{\infty} k^{p^2-2}$ *converges.*

Example 8.2.7 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2^k}{k2^k}\right)$ *converges or not.*

COMPARISON TEST.

Theorem 8.2.8 *Suppose that* $a_k \geq 0$ *for* $k \geq N$*. Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if its sequence of partial sums* $\{s_n\}$ *is bounded, i.e., if and only if there exists a finite number* $M > 0$ *such that*

$$\left|\sum_{k=1}^{n} a_k\right| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Theorem 8.2.9 (**Comparison Test**) *Suppose that there is an* $M \in \mathbb{N}$ *such that*

$$0 \leq a_k \leq b_k \quad \text{for all } k \geq M.$$

1. *If* $\sum_{k=1}^{\infty} b_k < \infty$*, then* $\sum_{k=1}^{\infty} a_k < \infty$.
2. *If* $\sum_{k=1}^{\infty} a_k = \infty$*, then* $\sum_{k=1}^{\infty} b_k = \infty$.

Example 8.2.10 *Determine whether the following series converges or diverges.*

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+3^k}$

Example 8.2.11 *Determine whether* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ *converges or diverges.*

LIMIT COMPARISON TEST.

Theorem 8.2.12 (**Limit Comparison Test**) *Suppose that a_k and b_k are positive for lagre k and*

$$L := \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists as an extended real number.

1. *If $0 < L < \infty$, then $\sum_{k=1}^{\infty} b_k$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
3. *If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Example 8.2.13 *Use the Limit Comparison Test to prove that* $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *converge.*

Example 8.2.14 *Determine whether* $\sum_{k=1}^{\infty} \frac{k}{2k^4+k+3}$ *converges or diverges.*

Example 8.2.15 *Determine whether* $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ *converges or diverges.*

Theorem 8.2.16 *Let* $a_k \to 0$ *as* $k \to \infty$. *Prove that*

$$\sum_{k=1}^{\infty} \sin|a_k| \text{ converges} \quad \text{if and only if} \quad \sum_{k=1}^{\infty} |a_k| \text{ converges.}$$

Exercises 8.2

1. Prove that each of the following series converges.

1.1 $\sum_{k=1}^{\infty} \frac{k-3}{k^3+k+1}$

1.2 $\sum_{k=1}^{\infty} \frac{k-1}{k2^k}$

1.3 $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}, \quad p > 1$

1.4 $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}3^{k-1}}$

1.5 $\sum_{k=1}^{\infty} \left(10 + \frac{1}{k}\right) k^{-e}$

1.6 $\sum_{k=1}^{\infty} \frac{3k^2 - \sqrt{k}}{k^4 - k^2 + 1}$

2. Prove that each of the following series diverges.

2.1 $\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$

2.2 $\sum_{k=1}^{\infty} \frac{1}{\ln^p(k+1)}, \quad p > 0$

2.3 $\sum_{k=1}^{\infty} \frac{k^2+2k+3}{k^3-2k^2+\sqrt{2}}$

2.4 $\sum_{k=1}^{\infty} \frac{1}{k\ln^p k}, \quad p \leq 1$

3. Use the Comparison Test to determine whether $\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\ln k}{k}}$ converges or diverges.

4. Find all $p \geq 0$ such that the following series converges. $\sum_{k=1}^{\infty} \frac{1}{k\ln^p(k+1)}$

5. If $a_k \geq 0$ is a bounded sequence, prove that $\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$ converges for all $p > 1$.

6. If $\sum_{k=1}^{\infty} |a_k|$ converges, prove that $\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$ converges for all $p \geq 0$. What happen if $p < 0$?

7. Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ coverge, then $\sum_{k=1}^{\infty} a_k b_k$ also converges.

8. Suppose tha $a, b \in \mathbb{R}$ satisfy $\frac{b}{a} \in \mathbb{R}\backslash\mathbb{Z}$. Find all $q > 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)q^k} \qquad \text{converges.}$$

9. Suppose that $a_k \to 0$. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty}(a_{2k} + a_{2k+1})$ converges.

8.3 Absolute convergence

Theorem 8.3.1 (**Cauchy Criterion**) *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$*, there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \textit{imply} \quad \left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Corollary 8.3.2 *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$*, there is an* $N \in \mathbb{N}$ *such that*

$$n \geq N \quad \textit{implies} \quad \left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon.$$

ABSOLUTE CONVERGENCE.

Definition 8.3.3 *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series.*

1. *S is said to* ***converge absolutely*** *if and only if* $\sum_{k=1}^{\infty} |a_k| < \infty$.
2. *S is said to* ***converge conditionally*** *if and only if S converges but not absolutely.*

Theorem 8.3.4 *A series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely if and only if for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Theorem 8.3.5 *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Example 8.3.6 *Prove that* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ *converges absolutely but* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *is not.*

LIMIT SUPREMUM.

Definition 8.3.7 *The supremum s of the set of adherent points of a sequence $\{x_k\}$ is called the* ***limit supremum*** *of $\{x_k\}$, denoted by $s := \limsup_{k\to\infty} x_k$, i.e.,*

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup\{x_k : k \geq n\}.$$

Example 8.3.8 *Evaluate limit supremum of the following sequences.*

1. $x_k = \frac{1}{k}$
2. $y_k = \frac{(-1)^k}{k}$
3. $z_k = 1 + (-1)^k$

Theorem 8.3.9 *Let* $x \in \mathbb{R}$ *and* $\{x_k\}$ *be a real sequence.*

1. *If* $\limsup_{k\to\infty} x_k < x$*, then* $x_k < x$ *for large* k*.*
2. *If* $\limsup_{k\to\infty} x_k > x$*, then* $x_k > x$ *for infinitely many* k*.*

Theorem 8.3.10 *Let* $x \in \mathbb{R}$ *and* $\{x_k\}$ *be a real sequence. If* $x_k \to x$ *as* $k \to \infty$*, then*

$$\limsup_{k\to\infty} x_k = x.$$

Example 8.3.11 *Evaluate limit supremum of* $\left\{\dfrac{k}{k+1}\right\}$.

ROOT TEST.

Theorem 8.3.12 (**Root Test**) *Let* $a_k \in \mathbb{R}$ *and* $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. *If* $r < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*
2. *If* $r > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Example 8.3.13 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{1+2k}\right)^k$ *converges absolutely.*

Example 8.3.14 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3+(-1)^k}{2}\right)^k$ *diverges.*

RATIO TEST.

Theorem 8.3.15 (**Ratio Test**) *Let* $a_k \in \mathbb{R}$ *with* $a_k \neq 0$ *for large* k *and suppose that*

$$r := \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists as an extended real number.

1. *If* $r < 1$*, then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*
2. *If* $r > 1$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Example 8.3.16 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$ *converges absolutely.*

Example 8.3.17 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ *diverges.*

Exercises 8.3

1. Prove that each of the following series converges.

1.1 $\sum_{k=1}^{\infty} \frac{1}{k!}$ 1.2 $\sum_{k=1}^{\infty} \frac{1}{k^k}$ 1.3 $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ 1.4 $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

2. Decide, using results convered so far in this chapter, which of the following series converge and which diverge.

2.1 $\sum_{k=1}^{\infty} \frac{k^2}{\pi^k}$ 2.4 $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+3}\right)^{k}$ 2.7 $\sum_{k=1}^{\infty} \left(\frac{k!}{(k+2)!}\right)^{k^2}$

2.2 $\sum_{k=1}^{\infty} \frac{k!}{2^k}$ 2.5 $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$ 2.8 $\sum_{k=1}^{\infty} \left(\frac{3+(-1)^k}{3}\right)^{k}$

2.3 $\sum_{k=1}^{\infty} \frac{k!}{2^k+3^k}$ 2.6 $\sum_{k=1}^{\infty} \left(\pi - \frac{1}{k}\right) k^{-1}$ 2.9 $\sum_{k=1}^{\infty} \frac{(1+(-1)^k)^k}{e^k}$

3. Define a_k recursively by $a_1 = 1$ and

$$a_k = (-1)^k \left(1 + k\sin\left(\frac{1}{k}\right)\right)^{-1} a_{k-1}, \quad k > 1.$$

Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

4. Suppose that $a_k \geq 0$ and $\sqrt[k]{a_k} \to a$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} a_k x^k$ converges absolutely for all $|x| < \frac{1}{a}$ if $a \neq 0$ and for all $x \in \mathbb{R}$ if $a = 0$.

5. For eachof the following, find all values of $p \in \mathbb{R}$ for which the given series converges absolutely.

5.1 $\sum_{k=2}^{\infty} \frac{1}{k \ln^p k}$ 5.3 $\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ 5.5 $\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k}$

5.2 $\sum_{k=2}^{\infty} \frac{1}{\ln^p k}$ 5.4 $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k^p-1)}$ 5.6 $\sum_{k=1}^{\infty} (\sqrt{k^{2p}+1} - k^p)$

6. Suppose that $a_{kj} \geq 0$ for $k, j \in \mathbb{N}$. Set

$$A_k = \sum_{j=1}^{\infty} a_{kj}$$

for each $k \in \mathbb{N}$, and suppose that $\sum_{k=1}^{\infty} A_k$ converges.

   6.1 Prove that $\displaystyle \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right)$

   6.2 Show that $\displaystyle \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{kj}\right) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{kj}\right)$

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove that $\sum_{k=1}^{\infty} |a_k|^p$ converges for all $p \geq 1$.

8. Suppose that $\sum_{k=1}^{\infty} a_k$ converges conditionally. Prove that $\sum_{k=1}^{\infty} k^p a_k$ diverges for all $p \geq 1$.

9. Let $a_n > 0$ for $n \in \mathbb{N}$. Set $b_1 = 0$, $b_2 = \ln\left(\frac{a_2}{a_1}\right)$, and

$$b_k = \ln\left(\frac{a_k}{a_{k-1}}\right) - \ln\left(\frac{a_{k-1}}{a_{k-2}}\right), \quad k = 3, 4, ...$$

   9.1 Prove that $r = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ if exists and is positive, then

$$\lim_{n\to\infty} \ln(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sum_{k=1}^{\infty}\left(1 - \frac{k-1}{n}\right) b_k = \sum_{k=1}^{\infty} b_k = \ln r.$$

   9.2 Prove that if $a_n \in \mathbb{R}\backslash\{0\}$ and $\left|\frac{a_{n+1}}{a_n}\right| \to r$ as $n \to \infty$, for some $r > 0$, then $|a_n|^{\frac{1}{n}} \to r$ as $n \to \infty$.

8.4 Alternating series

Theorem 8.4.1 (**Abel's Formula**) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *and* $\{b_k\}_{k\in\mathbb{N}}$ *be real sequences, and for each pair of integers* $n \geq m \geq 1$ *set*

$$A_{n,m} := \sum_{k=m}^{n} a_k.$$

Then

$$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$$

for all integers $n > m \geq 1$.

Theorem 8.4.2 (**Dirichilet's Test**) *Let* $\{a_k\}$ *and* $\{b_k\}$ *be sequences in* $\mathbb{R}$. *If the sequence of partial sums* $s_n = \sum_{k=1}^{n} a_k$ *is bounded and* $b_k \downarrow 0$ *as* $k \to \infty$, *then*

$$\sum_{k=1}^{n} a_k b_k \quad \textit{converges.}$$

Corollary 8.4.3 (**Alternating Series Test (AST)**) *If $a_k \downarrow 0$ as $k \to \infty$, then*

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \textit{converges.}$$

Moreover, if $\sum_{k=1}^{\infty} a_k$ *converges, then*

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \textit{converges conditionally.}$$

Example 8.4.4 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally.*

Example 8.4.5 *Prove that* $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ *converges conditionally.*

Example 8.4.6 *Prove that* $S(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ *converges for each* $x \in \mathbb{R}$.

Exercises 8.4

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left(\frac{\pi}{2}-\arctan k\right)$

1.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{2^k}$

1.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}$

1.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^p}, \quad p>0$

1.5 $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^p}, \quad x\in\mathbb{R},\ p>0$

1.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\frac{2\cdot 4\cdots(2k)}{1\cdot 3\cdots(2k-1)}$

1.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\ln(e^k+1)}$

1.8 $\displaystyle\sum_{k=1}^{\infty}\frac{\arctan k}{4k^3-1}$

2. For each of the following, find all values $x\in\mathbb{R}$ for which the given series converges.

2.1 $\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$

2.2 $\displaystyle\sum_{k=1}^{\infty}\frac{x^{3k}}{2^k}$

2.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{\sqrt{k^2+1}}$

2.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(x+2)^k}{k\sqrt{k+1}}$

2.5 $\displaystyle\sum_{k=1}^{\infty}\frac{2^k(x+1)^k}{k!}$

2.6 $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k(x+3)}{\cos k}\right)^k$

3. Using any test coveredin this chapter, find out which of the following series converge absolutely, which converge conditionally, and which diverge.

3.1 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^3}{(k+1)!}$

3.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)(-3)\cdots(1-2k)}{1\cdot 4\cdots(3k-2)}$

3.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(k+1)^k}{p^k k!}, \quad p>e$

3.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k+1}$

3.5 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k+1}}{\sqrt{k}k^k}$

3.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sin k}{k!}$

3.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}$

3.8 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\ln(k+2)}{k}$

4. **ABEL'S TEST.** Suppose that $\sum_{k=1}^{\infty} a_k$ converges and $b_k \downarrow b$ as $k \to \infty$. Prove that

$$\sum_{k=1}^{\infty} a_k b_k \quad \text{converges.}$$

5. Use Dirichilet's Test to prove that

$$S(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges for all $x \in \mathbb{R}$.

6. Prove that $\sum_{k=1}^{\infty} a_k \cos(kx)$ converges for every $x \in (0, 2\pi)$ and every $a_k \downarrow 0$.
What happens when $x = 0$?

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Prove that if $b_k \uparrow \infty$ and $\sum_{k=1}^{\infty} a_k b_k$ converges, then

$$b_m \sum_{k=m}^{\infty} a_k \to 0 \quad \text{as} \quad m \to \infty.$$

Reference

Gerald B. Folland. (1999). **Real Analysis Modern Technique and Their Applications**. John Wiley & Sons, Inc., New York.

Halsey L. Royden and Prtrick M. Fitzpatrick. (2010). **Real Analysis** (Fourth Edition). Pearson Education, Inc. New Jersey.

Pual Glendinning. (2012). **Maths in minutes**. Quercus Editions Ltd, London, England.

William R. Wade. (2004). **An Introduction Analysis** (Third Edition). Pearson Education. Inc., New Jersey.

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